MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems—Fall 2003

PROBLEM SET 2 SOLUTIONS

(E1) O&W 1.38(a)

(a) From **Figure 1.34** in O&W, we have the following

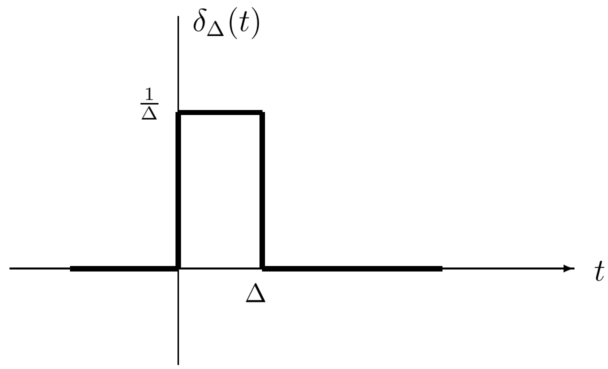


Figure 2.O1.1: Narrow Pulse

Now, consider $\delta_\Delta(2t)$ which is a time compressed version of **Figure 1.34** in Oppenheim and Willsky.

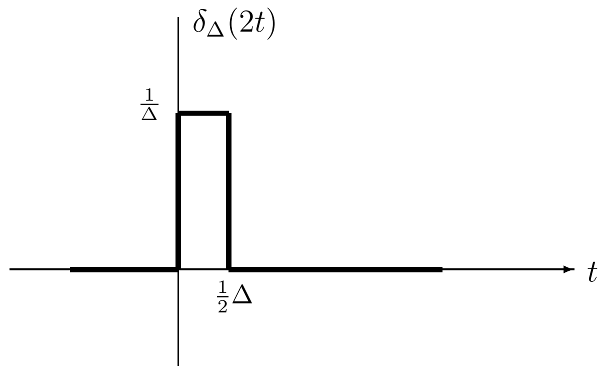


Figure 2.O1.2: Time Compressed Narrow Pulse

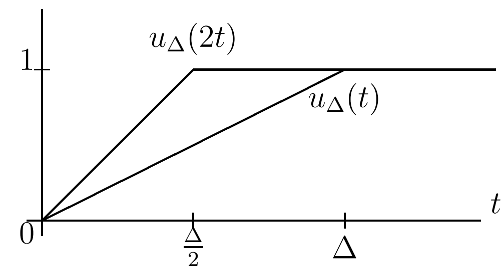
The area under this new pulse is of course $\frac{1}{2}$. If we take the limit as $\Delta \to 0$, we end up with:

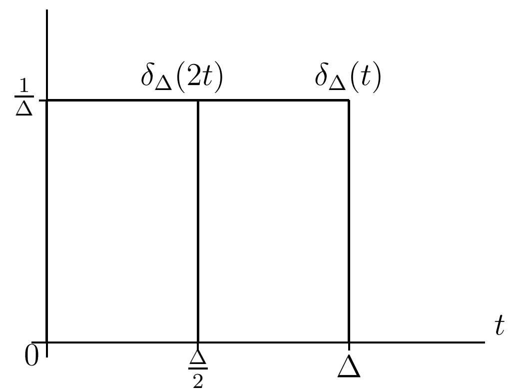
$$\delta(2t) = \lim_{\Delta \to 0} \delta_\Delta(2t)$$

The area under which remains $\frac{1}{2}$. From the result in **Section 1.4.2** of the text we have:

$$\delta(2t) = \frac{1}{2}\delta(t)$$

We can also show this using the relationship between the unit step function and the unit impulse (i.e. the unit impulse is the time derivative of the unit step function). For a given Δ, the approximation of both unit steps $u_\Delta(t)$ and $u_\Delta(2t)$ are shown to the right. Note that $u_\Delta(2t)$ reaches unity at $t = \frac{\Delta}{2}$.





Making a change of variable $s = 2t$ where $ds = 2dt$ and using Eqn(1.73) in O&W, gives

$$\begin{aligned} \delta_\Delta(s) &= \frac{du_\Delta(s)}{ds} = \frac{du_\Delta(2t)}{2dt} \\ &= \frac{1}{2}\frac{du_\Delta(2t)}{dt} = \frac{1}{2}\frac{2}{\Delta} = \frac{1}{\Delta}. \end{aligned}$$

The area enclosed by $\delta_\Delta(2t)$ is half of that of $\delta_\Delta(t)$ as shown to the right. Since $\delta(t)$ is defined by it's area, we get $\delta(2t) = \frac{1}{2}\delta(t)$. In general

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

holds for any nonzero real number a. The above is a proof of the time scaling property, which tells us that we've simply squeezed $\delta_\Delta(t)$ by a factor of two.

(E2) O&W 2.33 (a-(i))

(a) (i) We need to find the homogeneous and particular solutions with the final solution being the sum of the two. Let's start with the particular solution which we will denote $y_p(t)$. Let the particular solution take the form of the input for $t > 0$, $y_p(t) = A\mathrm{e}^{3t}$. Substituting into $(P2.33 - 1)$ for $t > 0$ we have,

$$\begin{aligned} 3A\mathrm{e}^{3t} + 2A\mathrm{e}^{3t} &= \mathrm{e}^{3t} \\ A &= \frac{1}{5} \end{aligned}$$

Which gives us the particular solution, $y_p(t) = \frac{1}{5}\mathrm{e}^{3t}$.

For the homogeneous part, let's try a general exponential, again for $t > 0$. Denoting the homogeneous solution as $y_h(t)$, we have $y_h(t) = B\mathrm{e}^{st}$ where B and s are constants to be determined. Substituting into $(P2.33 - 1)$ with the input set to 0 we have,

$$\begin{aligned} Bs\mathrm{e}^{3t} + 2B\mathrm{e}^{3t} &= 0 \\ s + 2 &= 0 \\ s &= -2 \end{aligned}$$

Thus the homogeneous solution takes the form $B\mathrm{e}^{-2t}$. The output is then given by

$$\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= B\mathrm{e}^{-2t} + \frac{1}{5}\mathrm{e}^{3t} \end{aligned}$$

We know that the system is initially at rest, so at $t = 0$, the output has to be zero:

$$\begin{aligned} y(t) &= B\mathrm{e}^{-2(0)} + \frac{1}{5}\mathrm{e}^{3t} \\ B &= -\frac{1}{5} \end{aligned}$$

Thus, the output, $y(t) = -\frac{1}{5}\mathrm{e}^{-2t} + \frac{1}{5}\mathrm{e}^{3t}$.

Note that we had to solve for B with the additional information (in addition to the differential equation) that the system was at initial rest. This is because LCCDEs are **NOT** complete characterizations of systems. In general, we need more information to compute the output when given and input signal.

(E3) O&W 2.44 (a)

(a) Consider the general signals that satisfy the given restrictions as depicted below. Note, that the graph is of the time-flipped version of $h(\tau)$.

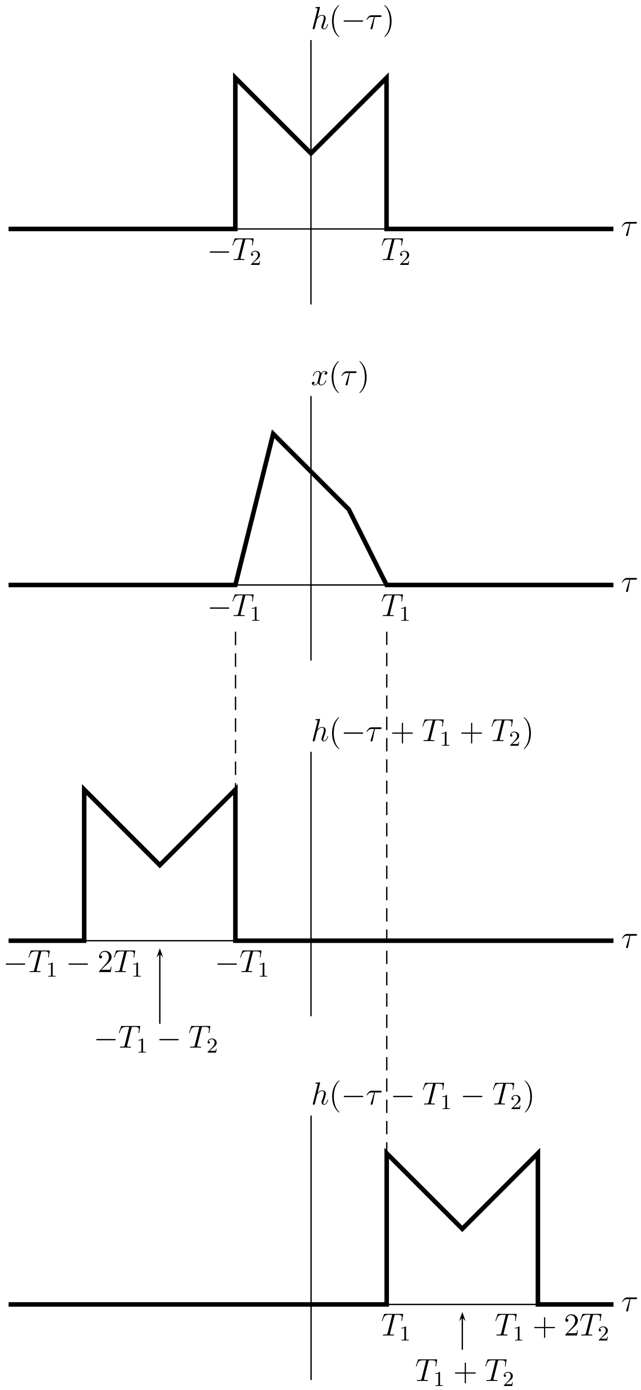


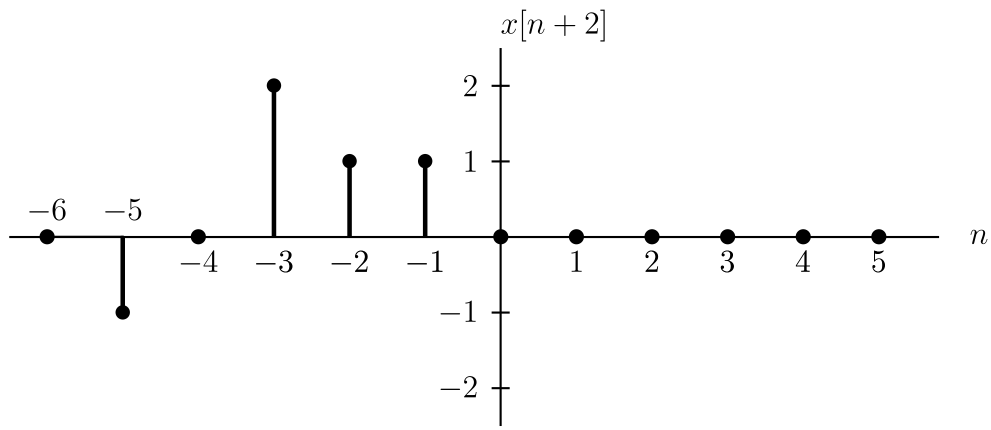
Figure 2.03: x(t) and h(t) at Break Points in Flip and Slide

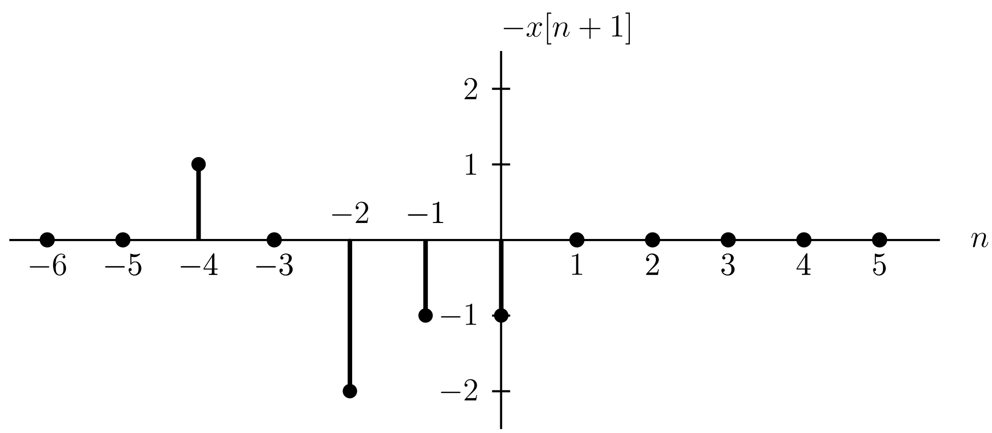
Using the flip and slide method to perform the convolution, we see that the product of the two signals (based on overlap) is first non-zero at $t = -T_1 - T_2$ and it stops being non-zero at $t = T_1 + T_2$. Thus, the output, $y(t)$ is zero for $|t| > T_1 + T_2$.

Generally, when convolving two CT signals of finite duration, the result starts being non-zero at the sum of the time indices when the two original signals start being non-zero. Similarly, the result of the convolution ends being non-zero at the sum of the time indices when the signals to be convolved end being non-zero.

Problem 1

(a) With short duration DT sequences, it is often simplest to find their convolution by centering copies of one of the signals about each of the non-zero samples of the other signal and scaled by the value of the sample at that location. The result is the sum of all the shifted and scaled signals. Thus, $y[n]$ is given by the sum of the following signals.





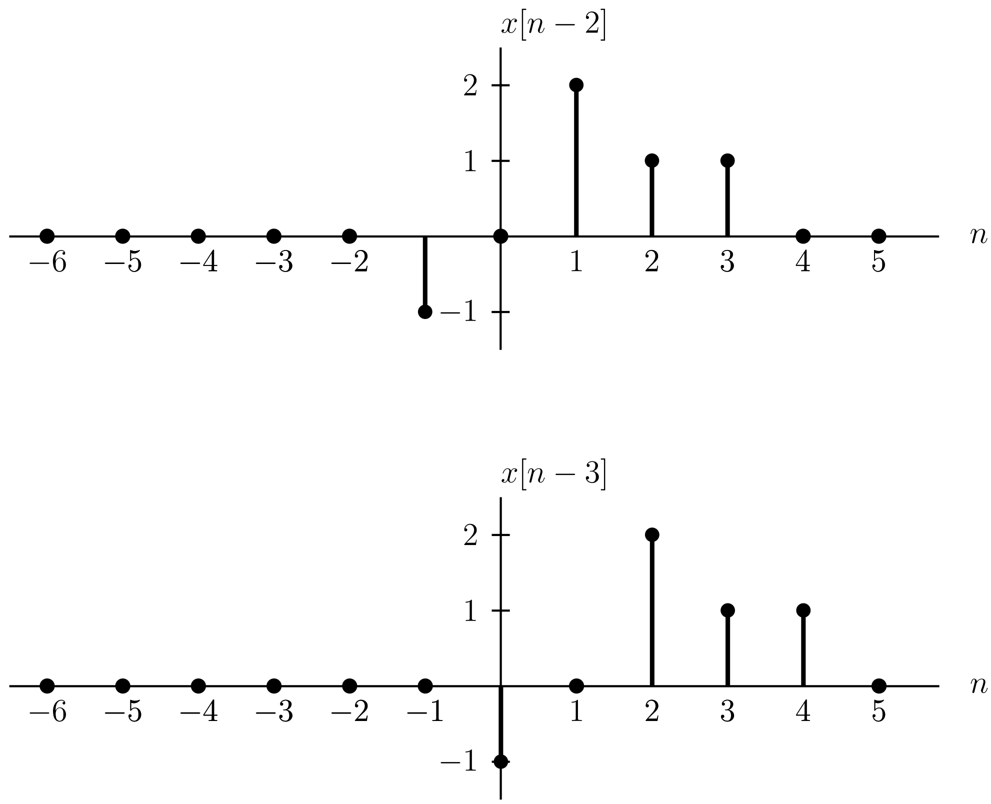


Figure 2.1.a.1: $x[n]$ **Scaled and shifted**

The sum of these yields the following sequence for $y[n]$:

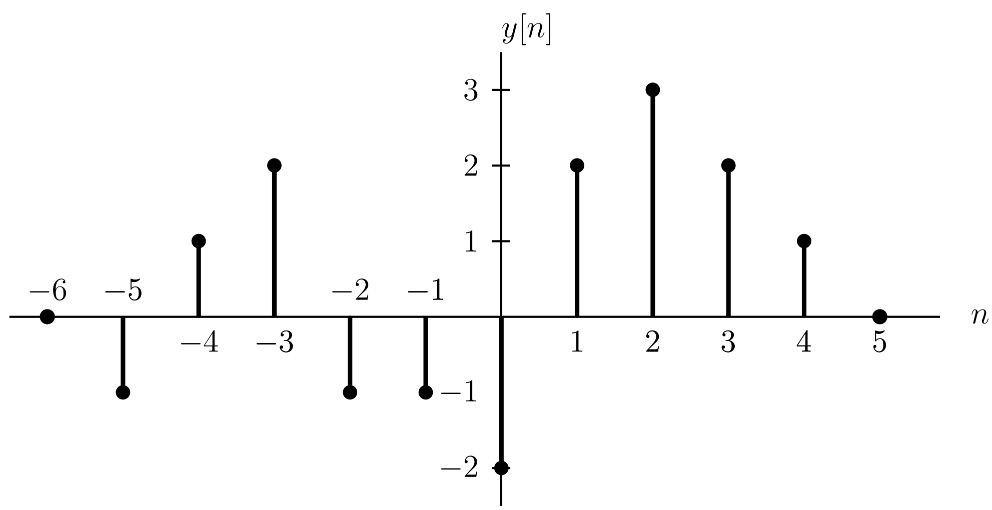


Figure 2.1.a.2: $y[n]$

(b) For this part, we can again use the shift and scale method since the sequence $x[n]$ is of a short duration as given below:

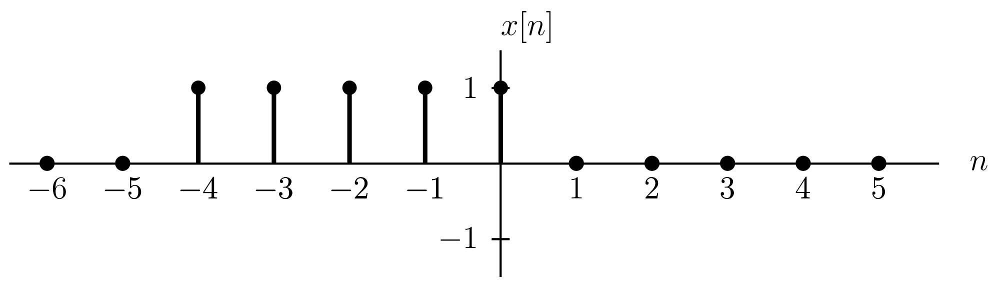


Figure 2.1.b: $x[n]$

Thus, we can write the output as a sum of scaled shifted inputs as follows:

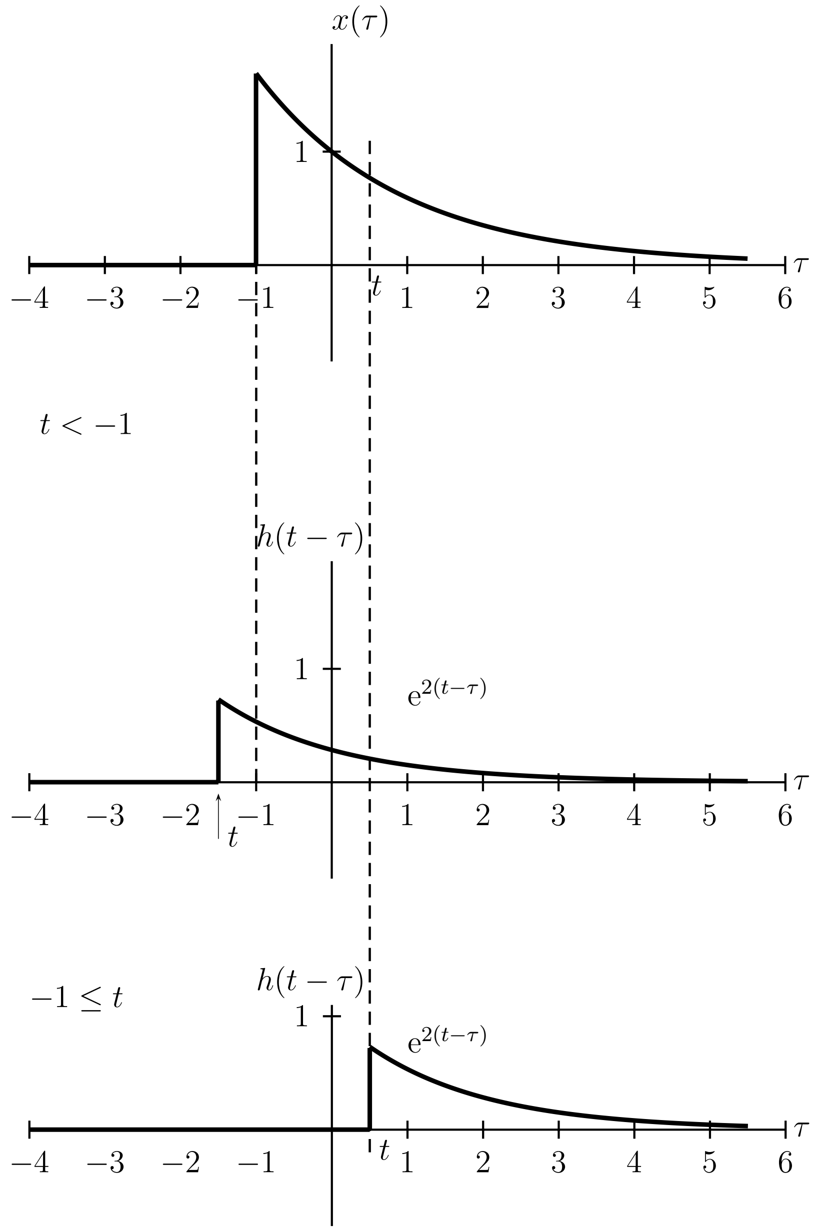
$$
\begin{aligned}
y[n] &= 2^n u[2-n] + 2^{n+1}u[1-n] + 2^{n+2}u[-n] + 2^{n+3}u[-n-1] + 2^{n+4}u[-n-2] \\
&= \sum_{k=0}^{4} 2^{n+k} u[2-n-k]
\end{aligned}
$$

Problem 2

(a) From the definition of the convolution, we have the following expression for the output $y(t)$:

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

Based on the given $x(t)$ and $h(t)$, we can break the integration up into 2 regions as illustrated in the diagram. The ranges are $t < -1$ and $t \geq -1$.



For the range $t < -1$, the region where $x(\tau)h(t-\tau)$ is non-zero is from $-1 \rightarrow \infty$. So, the expression for $y(t)$ is given by:

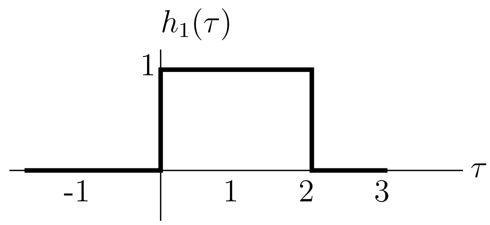
$$
\begin{aligned}
y(t) &= \int_{-1}^{\infty} h(t-\tau)x(\tau)d\tau = \int_{-1}^{\infty} \mathrm{e}^{2(t-\tau)}\mathrm{e}^{-\tau}d\tau \\
&= \mathrm{e}^{2t}\int_{-1}^{\infty} \mathrm{e}^{-3\tau}d\tau = \mathrm{e}^{2t}\left[-\frac{1}{3}\mathrm{e}^{-3\tau}\right]_{-1}^{\infty} \\
&= \frac{1}{3}\mathrm{e}^{2t+3}
\end{aligned}
$$

For the range $t \geq -1$, the $x(\tau)h(t-\tau)$ is non-zero for $\tau > t$. So the expression for $y(t)$ is given by:

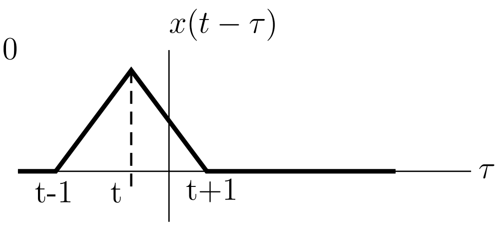
$$
\begin{aligned}
y(t) &= \int_{t}^{\infty} h(t-\tau)x(\tau)d\tau = \int_{t}^{\infty} \mathrm{e}^{2(t-\tau)}\mathrm{e}^{-\tau}d\tau \\
&= \mathrm{e}^{2t}\int_{t}^{\infty} \mathrm{e}^{-3\tau}d\tau = \mathrm{e}^{2t}\left[-\frac{1}{3}\mathrm{e}^{-3\tau}\right]_{t}^{\infty} \\
&= \mathrm{e}^{2t}\left[-\frac{1}{3}\mathrm{e}^{-3t}\right] \\
&= \frac{1}{3}\mathrm{e}^{-t}
\end{aligned}
$$

(b) Here, we can break $h(t)$ up into $h(t) = h_1(t) + h_2(t)$ where $h_1(t)$ is the "box" part of $h(t)$ and $h_2(t)$ are the two impulses. Let $y_1(t)$ and $y_2(t)$ denote the result of convolving $x(t)$ with $h_1(t)$ and $h_2(t)$ respectively.

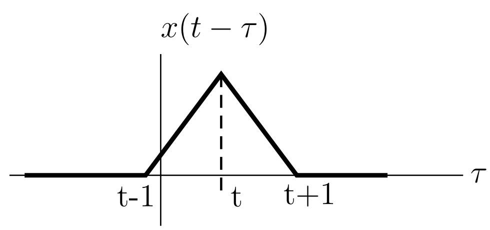
First let us compute $y_1(t)$. To do this, we fix $h_1(t)$ and flip and slide $x(t)$. The following figure illustrates the different regions of overlap.



$-1 < t < 0$



$0 \leq t < 1$



For the range $-1 < t < 0$, the result of the convolution is the area under the product of the two signals which is given by:

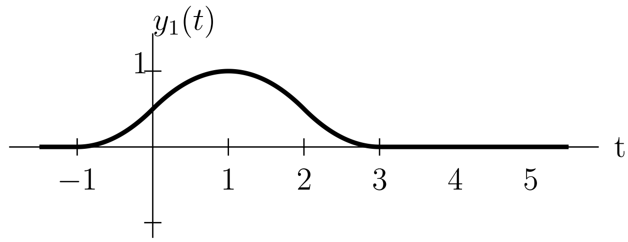
$$
\begin{aligned}
y_1(t) &= \frac{1}{2}(t+1)(t+1) \\
&= \frac{1}{2}(t^2 + 2t + 1)
\end{aligned}
$$

For the range $0 \leq t < 1$, the area under the product is given by:

$$
\begin{aligned}
y_1(t) &= t(1-t) + \frac{1}{2}t(1-(1-t)) + \frac{1}{2} \\
&= t - t^2 + \frac{1}{2}t^2 + \frac{1}{2} \\
&= \frac{1}{2}(1 + 2t - t^2)
\end{aligned}
$$

Now both $x(t)$ and $h_1(t)$ are symmetric signals which are symmetric about $t = 0$ and $t = 1$ respectively. Therefore, the convolution of the two is symmetric about $t = 1$.

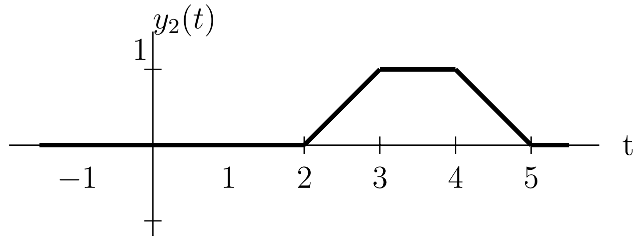
The plot for $y_1(t)$ looks like the following:



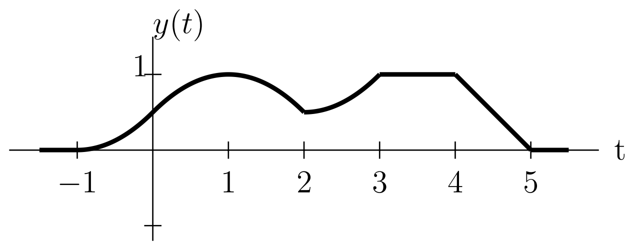
With the different regions of the curve as follows:

$$
\begin{aligned}
-1 < t < 0, &\qquad y_1(t) = \frac{1}{2}(t^2 + 2t + 1) \\
0 \le t < 1, &\qquad y_1(t) = \frac{1}{2}(1 + 2t - t^2) \\
1 \le t < 2, &\qquad y_1(t) = \frac{1}{2}(1 + 2t - t^2) \\
2 \le t < 3, &\qquad y_1(t) = \frac{1}{2}(t^2 - 6t + 9)
\end{aligned}
$$

The convolution with $h_2(t)$ is straightforward because it is a convolution with impulses. To do this, all we need is to center the triangle around both impulses and scale by the area under each impulse which in this case is 1. This gives the following plot for $y_2(t)$.



The final result is the sum of the two as follows:



The curved parts of the plot are given by the following expressions:

$$
\begin{aligned}
-1 < t < 0, \qquad & y(t) &= \frac{1}{2}(t^2 + 2t + 1) \\
0 \le t < 1, \qquad & y(t) &= \frac{1}{2}(1 + 2t - t^2) \\
1 \le t < 2, \qquad & y(t) &= \frac{1}{2}(1 + 2t - t^2) \\
2 \le t < 3, \qquad & y(t) &= \frac{1}{2}(t^2 - 4t + 5)
\end{aligned}
$$

Problem 3

(a) Since the unit sample response is non-zero for $n < 0$, the system is not causal. For stability, we need to ensure that the impulse response is absolutely summable.

$$\begin{aligned}\sum_{k=-\infty}^{\infty} |h[n]| &= \sum_{k=-\infty}^{3} 2^k \\ &= \sum_{k=\infty}^{-3} \left(\frac{1}{2}\right)^k\end{aligned}$$

which is finite. Thus, the system is stable

(b) Since $h(t)$ is 1 for $t < 0$, the system is not causal. For stability, the impulse response has to be absolutely integrable:

$$\begin{aligned}\int_{-\infty}^{\infty} |h(t)|dt &= \int_{-\infty}^{\infty} h(t)dt \qquad \text{since } h(t) \text{ is never negative} \\ &= \int_{-\infty}^{\infty} \left(u(1-t) - \frac{1}{2}\mathrm{e}^{-t}u(t)\right) dt \\ &= \int_{-\infty}^{0} u(1-t)dt + \int_{0}^{\infty} \left(u(1-t) - \frac{1}{2}\mathrm{e}^{-t}u(t)\right) dt \\ &= \int_{-\infty}^{0} 1dt + \int_{0}^{\infty} \left(u(1-t) - \frac{1}{2}\mathrm{e}^{-t}u(t)\right) dt\end{aligned}$$

The first term on the r.h.s. of the equation integrates to ∞ but the second term is finite, which means the sum of the two terms is infinite. So, the system is not stable.

(c) This system is causal because the impulse response is zero for $n < 0$. For stability, the impulse response has to be absolutely summable.

$$\begin{aligned}\sum_{k=-\infty}^{\infty} |h[k]| &= \sum_{k=0}^{\infty} h[k] + \sum_{k=-\infty}^{-1} -h[k] \\ &= \sum_{k=0}^{\infty} [1-(0.99)^k]u[k] + \sum_{k=-\infty}^{-1} [1-(0.99)^k]u[k] \\ &= \sum_{k=0}^{\infty} [1-(0.99)^k] \\ &= \sum_{k=0}^{\infty} 1 - \sum_{k=0}^{\infty} (0.99)^k\end{aligned}$$

The second term on the r.h.s. is finite, as we know from power series, and the formulae we derived in problem set 1. The first term on the r.h.s. of the equation is infinite. So, the r.h.s. is infinite, which means the system is not stable.

(d) Since $h(t) = 0$ for all $t < 0$, this system is causal. Now, let's check for stability by taking the integral of the absolute value.

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)| dt &= \int_{-\infty}^{\infty} h(t) dt \qquad \text{since } h(t) \text{ is always positive} \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{15t}[u(t-1) - u(t-100)] dt \\
&= \int_{1}^{100} \mathrm{e}^{15t} dt \\
&= \frac{1}{15}\mathrm{e}^{15t}\Big|_{1}^{100} \\
&= \frac{1}{15}(\mathrm{e}^{1500} - \mathrm{e}^{15})
\end{aligned}$$

Which is finite. So, the system is stable.

Problem 4

We can plug-in the values of the input into the difference equation to compute the output keeping in mind that the system is initially at rest.

We can rewrite the difference equation as:

$$y[n] = \frac{1}{2}y[n-1] + 2x[n] - x[n-2].$$

From the difference equation and the given input signal, we see that the first non-zero output occurs at $n = -2$ which is the first non-zero input. We can iterate through the time indices from $n = -2$ to $n = 5$. After that, we can write down an expression for the remaining samples because the input no longer drives the system after $n = 5$. At $n = 6, 7, 8 \cdots$, the output only depends on the previous output. Lets iterate through the first 8 output samples:

$$\begin{aligned}
y[-2] &= 0 + 2 \cdot 2 - 0 = 4 \\
y[-1] &= \left(\frac{1}{2}\right) \cdot 4 + 2 \cdot 1 - 0 = 4 \\
y[0] &= \left(\frac{1}{2}\right) \cdot 4 + 0 - 2 = 0 \\
y[1] &= 0 + 0 - 1 = -1 \\
y[2] &= -\left(\frac{1}{2}\right) \cdot 1 + 2 \cdot 1 - 0 = \frac{3}{2} \\
y[3] &= \left(\frac{1}{2}\right) \cdot \frac{3}{2} + 2 \cdot 1 - 0 = \frac{11}{4} \\
y[4] &= \left(\frac{1}{2}\right) \cdot \frac{11}{4} + 0 - 1 = \frac{3}{8} \\
y[5] &= \left(\frac{1}{2}\right) \cdot \frac{3}{8} - 1 = -\frac{13}{16}
\end{aligned}$$

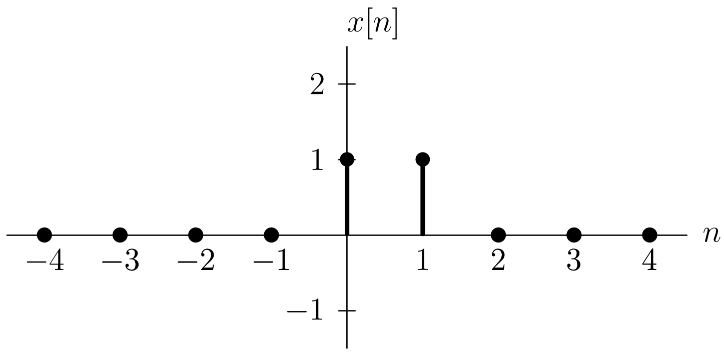
from $n = 6$ onward, the output is just one half of the previous output so, for $n \geq 6$, we have,

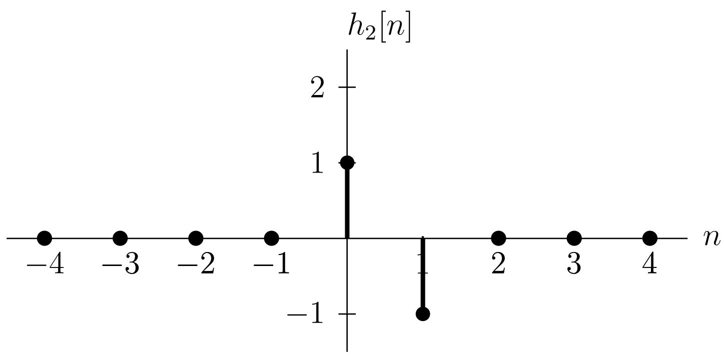
$$y[n] = -\left(\frac{1}{2}\right)^{n-5} \cdot \frac{13}{16}.$$

So, the complete solution is the above expression, the values computed by iteration and $y[n] = 0$ for $n < -2$ because of initial rest.

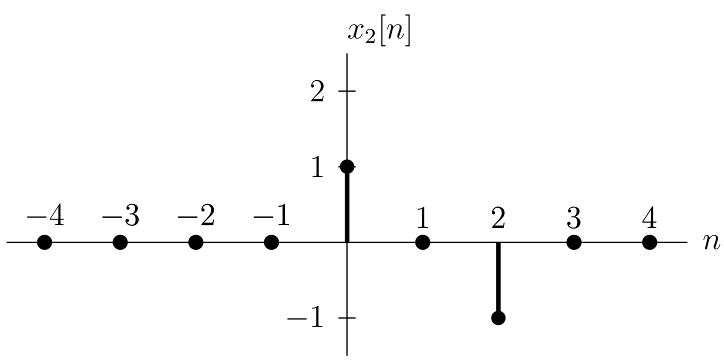
Problem 5

Since convolution is commutative, we can convolve $x[n]$ with $h_2[n]$ first followed by a convolution with $h_1[n]$ to get $y[n]$. Let's start by convolving $x[n]$ with $h_2[n]$ and denote the result of this as $x_2[n]$. $x[n]$and $h[n]$ are given by the following:

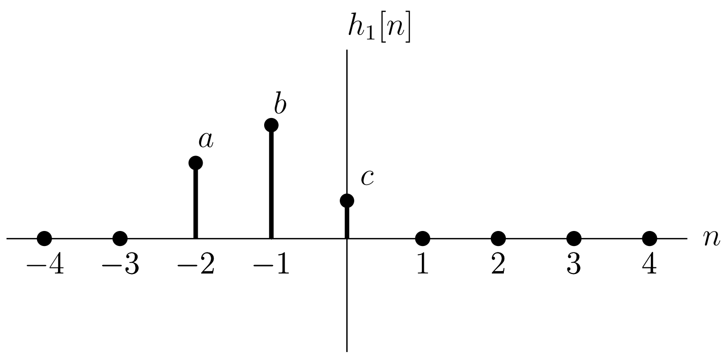




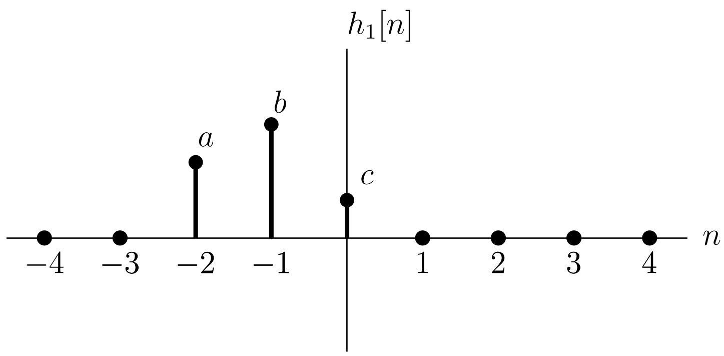
So, $x_2[n]$ is given by the following:

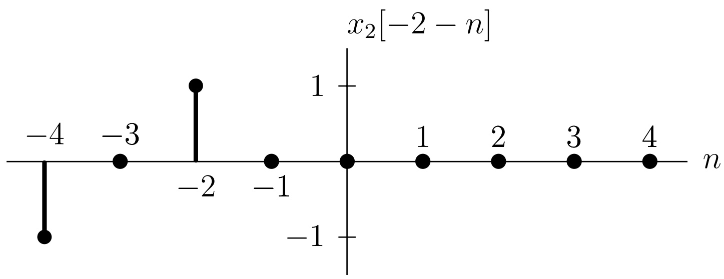


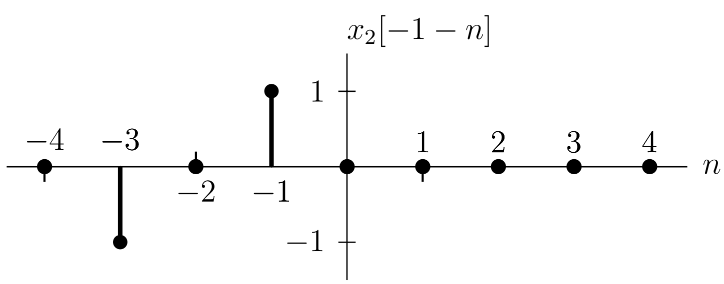
Now, we need to figure out the sequence $h_1[n]$ that when convolved with $x_2[n]$ produces $y[n]$. First, based on the starting and ending points of $y[n]$, we can determine the first and last non-zero points of $h_1[n]$. This is because the first non-zero point of $y[n]$ is at the time index that is the sum of the first non-zero indices of $x_2[n]$ and $h_1[n]$. Which means that the first non-zero point of $h_1[n]$ is at $n = -2$. Similarly, we know that the ending point is at the index $n = 0$ We can use flip and slide mechanics to determine the values of the samples between $n = -2$ and $n = 0$. Let the following be a general stem plot of $h_1[n]$:

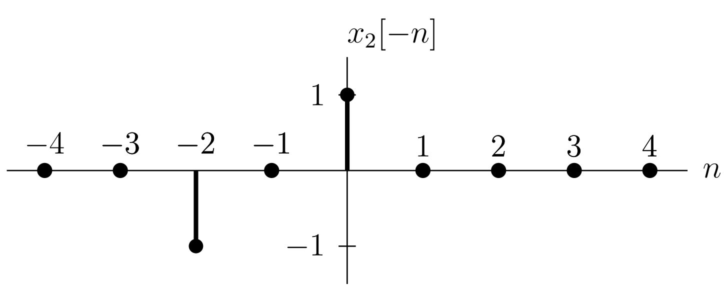


If we flip and slide $x_2[n]$ against $h_2[n]$ and compare against the given $y[n]$, we have the following:









From the plot of $x[-2-n]$ we find that:

$$\begin{aligned} y[-2] &= a+0 \\ a &= 2 \end{aligned}$$

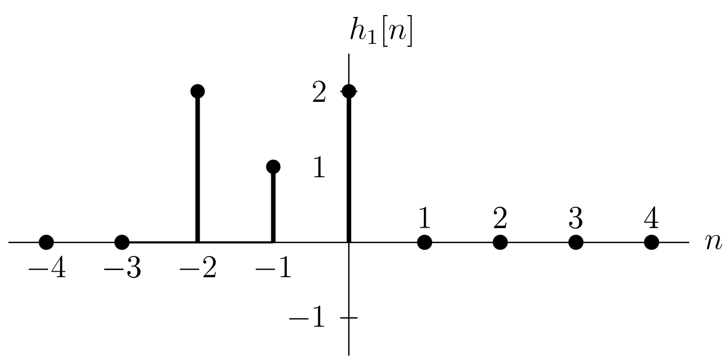
From the plot of $x[-1-n]$ we find that:

$$\begin{aligned} y[-1] &= b+0 \\ b &= 1 \end{aligned}$$

From the plot of $x[-n]$ we find that:

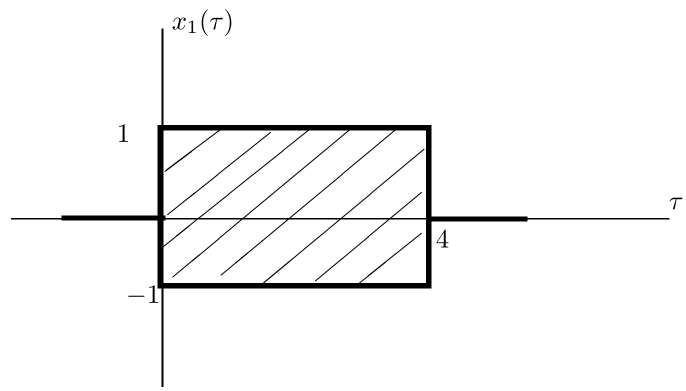
$$\begin{aligned} y[0] &= c-a \\ 0 &= c-2 \\ c &= 2 \end{aligned}$$

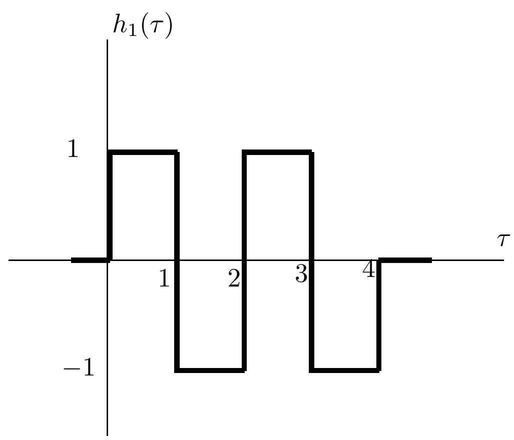
Thus, we have found the entire sequence $h_1[n]$ which is given as follows:

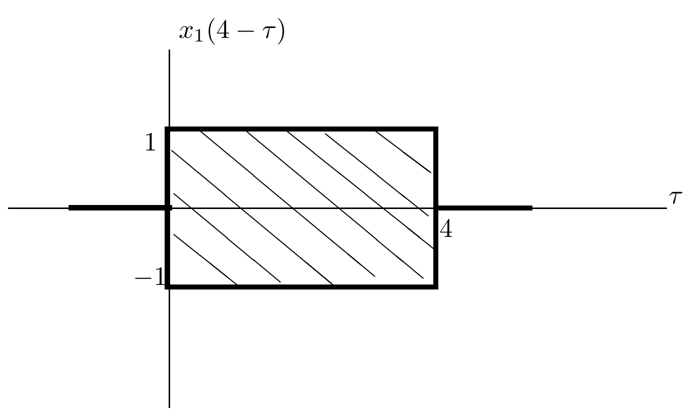


Problem 6

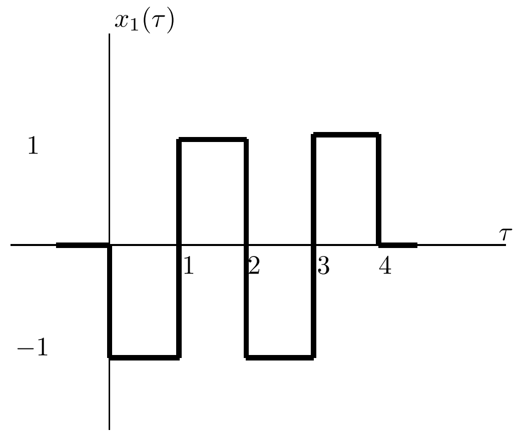
(a) Consider the flip and slide operation to find the result of the convolution of $x_1(t)$ and $h_1(t)$ at $t = 4$. If we fix $h_1(\tau)$ and did the flip and slide on $x_1(\tau)$, we have the following plot for some $x_1(t - \tau)$ (to be determined) and $h_1(\tau)$.



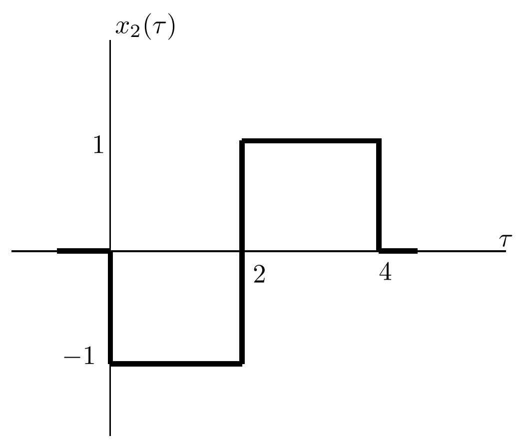




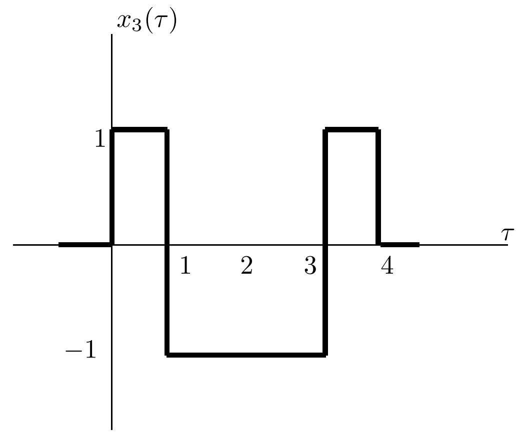
From this plot, it is clear that at $t = 4$, the signal $x_1(4 - \tau)$ that is identical to $h_1(\tau)$ will result in the largest area under the product of $x_1(4 - \tau)$ and $h_1(\tau)$. Since this will ensure that the negative portions of $h_1(\tau)$ are multiplied by negative values of $x_1(4 - \tau)$. If $x_1(4 - \tau)$ is identical to $h_1(\tau)$, then $x_1(\tau)$ is given by



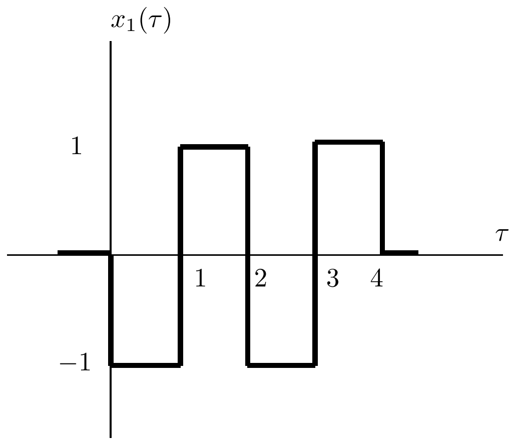
(b) Similarly, for the next two signals, we have the maximum value of the convolution at $t = 4$ given by flipped and shifted (by 4) versions of the input themselves. This yields the following for $x_2(t)$

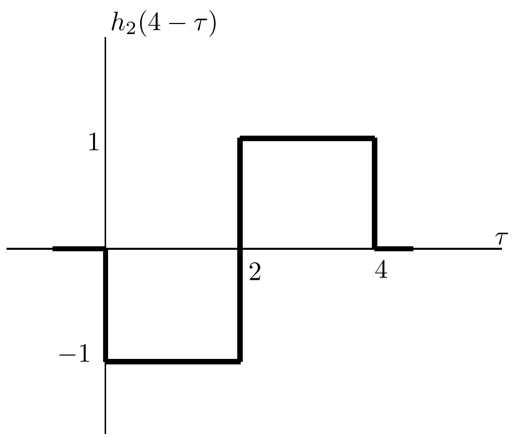


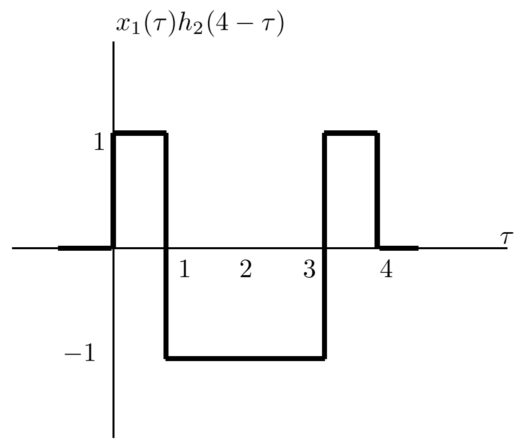
The signal $h_3(t)$ looks the same when it is flipped (about the $t = 2$ line) or not. So, the result is identical to the impulse response and is given by:

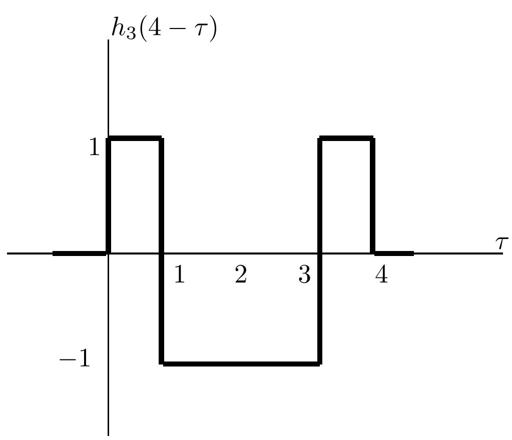


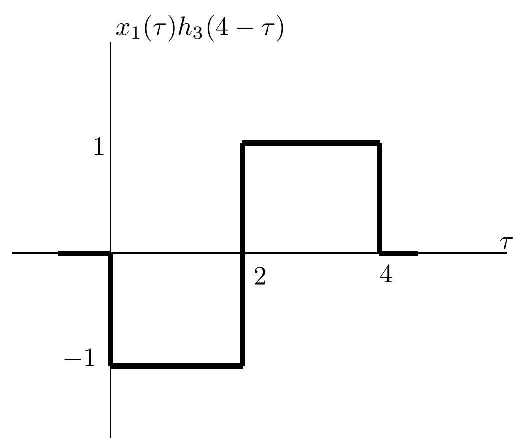
(c) The following plot illustrates the values of $x_1(t) * h_2(t)$ and $x_1(t) * h_3(t)$. It shows the signal $x_1(t)$ fixed with $h_2(t)$ and $h_3(t)$ flipped and slid to the appropriate location. Clearly, the area under the products are zero for both cases


$x_1(\tau)$
1
−1
τ
1
2
3
4

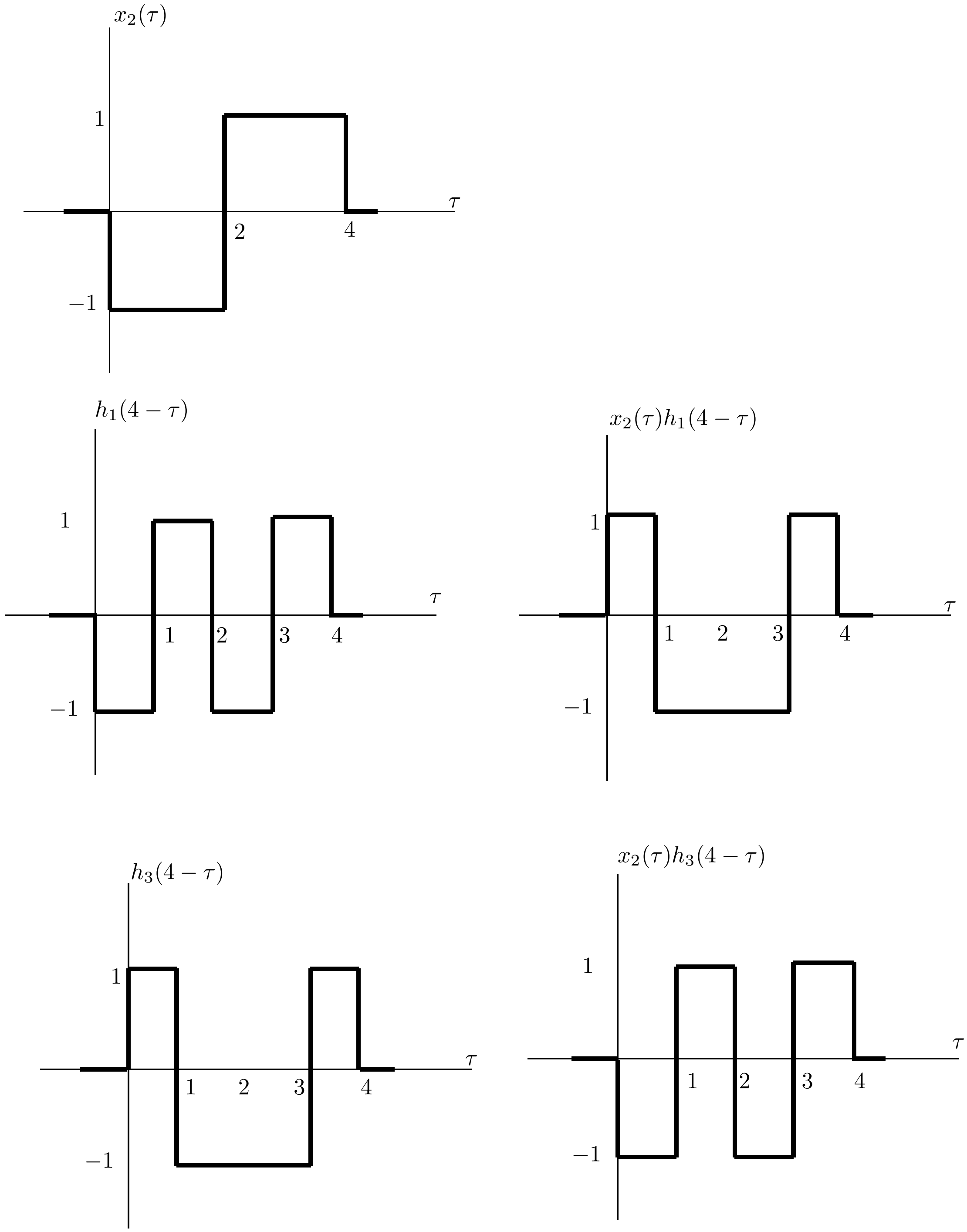

$h_2(4-\tau)$
1
−1
τ
2
4


$x_1(\tau)h_2(4-\tau)$
1
−1
τ
1
2
3
4

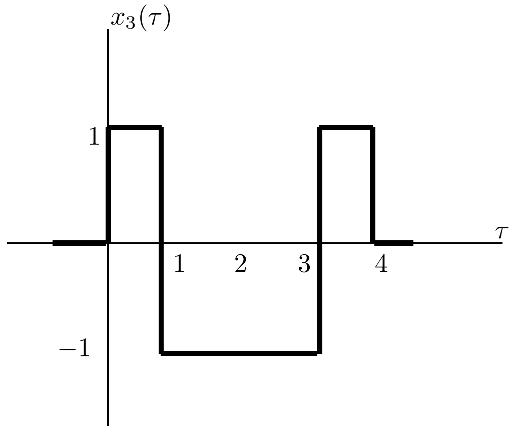

$h_3(4-\tau)$
1
−1
τ
1
2
3
4

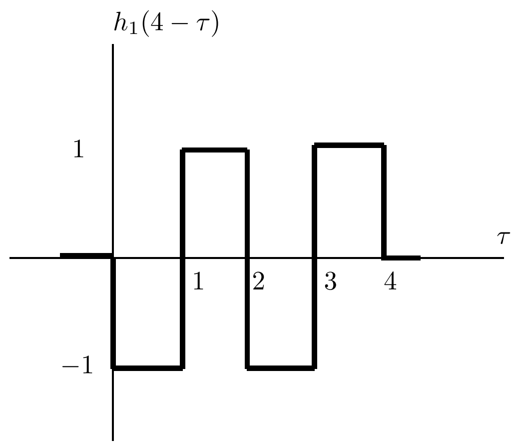

$x_1(\tau)h_3(4-\tau)$
1
−1
τ
2
4

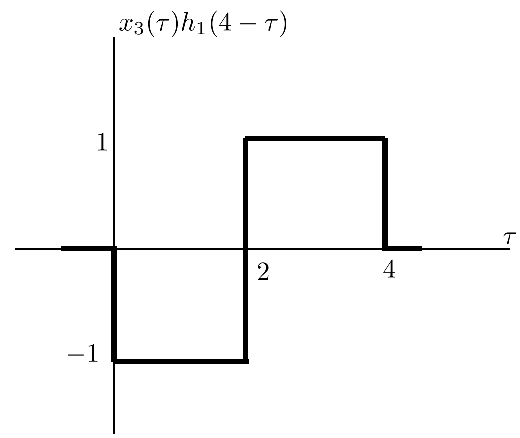
The next figure illustrates this for $x_2(t) * h_1(t)$ $x_2(t) * h_3(t)$. Again, the area under the products are zero for both cases.

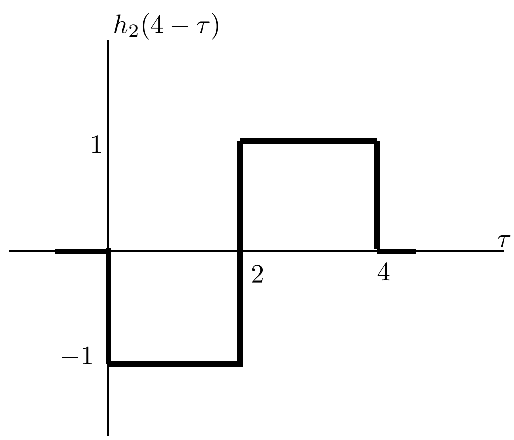


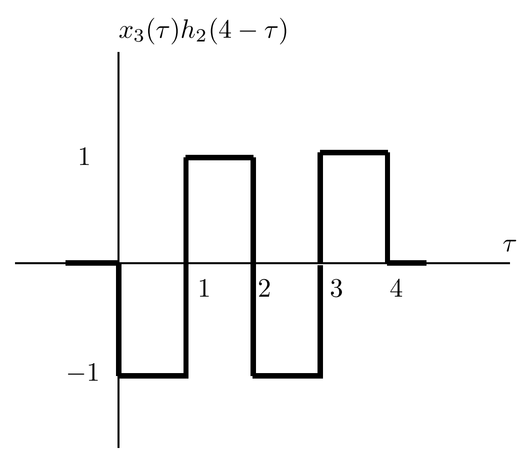
The next figure illustrates this for $x_3(t) * h_1(t)$ $x_3(t) * h_2(t)$. Again, the area under the products are zero for both cases.









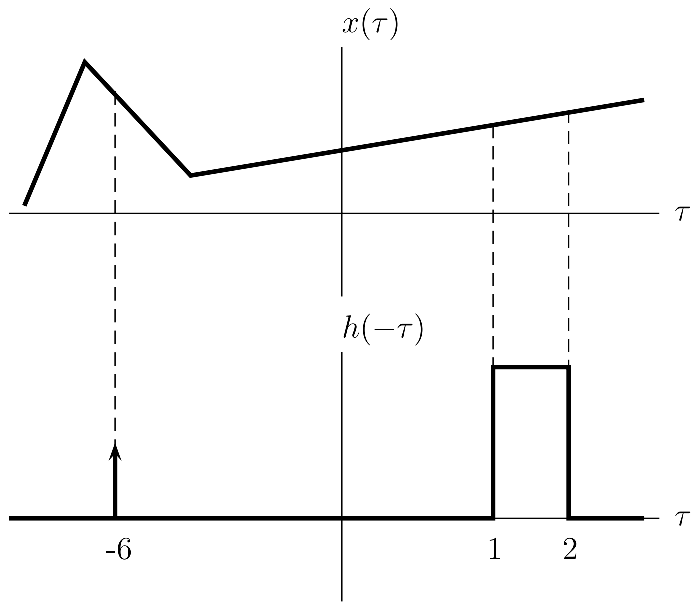


Problem 7

Consider a generic signal $x(\tau)$. The value of $y(0)$ can be found by evaluating the colvolution integral at $t = 0$.

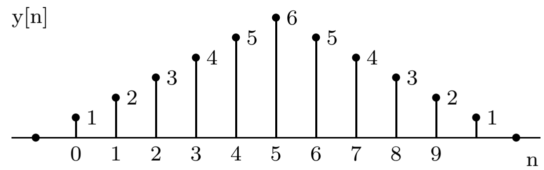
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$
$$y(t)|_{t=0} = \int_{-\infty}^{\infty} x(\tau)h(-\tau)d\tau$$

This is just the area under the product of the two signals depicted below. Since $h(-\tau)$ is zero everywhere except for $2 < \tau < 1$ and at $\tau = -6$, the product of the two signals will also be zero outside this range. Thus, we only need to know $x(\tau)$ at $1 < \tau < 2$ and $\tau = -6$.



Problem 8 (BDS 2.1)

(a) Given $x[n] = \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & \text{otherwise} \end{cases}$, we are to find $y[n] = x[n] * x[n]$. The easiest way to do this is to do it graphically. Flipping, multiplying, summing, and sliding $x[n]$ produces the output on the right.



(b) The BDS MATLAB workbook shows what the $y[n]$ plot obtained using MATLAB should look like, and an example of code that can produce that plot is:

```
x = ones(1,6);
y = conv(x, x);
ny = 0:10;
stem(ny,y)
```

(c) Once again, the BDS MATLAB workbook shows $y[n]$ obtained using MATLAB. Here is code that can generate the desired plot:

```
x = ones(1,6);
h = 0:5;
y = conv(x, h);
ny = 0:10;
stem(ny,y)
```

(d) Starting with $y_2[n] = x[n] * h[n+5]$, we can derive

$$\begin{aligned} y_2[n] &= x[n] * h[n+5] \\ &= x[n] * h[n] * \delta[n+5] \\ &= y[n] * \delta[n+5] \\ &= y[n+5]. \end{aligned}$$

(e) Once again, the BDS MATLAB workbook shows what $y_2[n]$ obtained using MATLAB should be. Here is code that can generate the desired plot:

```
x = ones(1,6);
h = 0:5;
y2 = conv(x, h);
ny2 = -5:5;
stem(ny2,y2)
```

Problem 9 (BDS 2.2)

(a) To define the causal LTI system given by $y[n] = 0.5x[n] + x[n-1] + 2x[n-2]$, you would define `a1` and `b1` as follows in MATLAB:

```
a1 = [1];
b1 = [0.5 1 2];
```

(b) To define the causal LTI system given by $y[n] = 0.8y[n-1] + 2x[n]$, you would define `a2` and `b2` as follows in MATLAB:

```
a2 = [1 -0.8];
b2 = [2];
```

(c) To define the causal LTI system given by $y[n] - 0.8y[n-1] = 2x[n-1]$, you would define `a3` and `b3` as follows in MATLAB:

```
a3 = [1 -0.8];
b3 = [0 2];
```

(d-f) The workbook provides solutions.

(g) To generate $x[n]$ from $0 \leq n \leq 10$ in MATLAB, you would use
`x2 = [ ones(1,6) zeros(1,5) ];`

(h) The values in h2 are the same as in h.

(i) The plot is the same as BDS Fig. 2.4 advanced by 5.

(j) This is just like part (g) except you define your time axis differently. The MATLAB code would be

```
x2 = [ ones(1,6) zeros(1,5) ];
h2 = 0:5;
y2 = filter(h2, 1, x2);
ny2 = -5:5;
stem(ny2,y2)
```