Rings of Regular Functions on Spherical Nilpotent Orbits for Complex Classical Groups

by

Tonghoon Suk

B.S., Mathematics and Physics (2001), Seoul National University

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Master of Science in Mathematics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 2009

Author ....
Department of Mathematics
August 7, 2009

Certified by ....
David Alexander Vogan
Professor of Mathematics
Thesis Supervisor

Accepted by..
David Jerison
Chairman, Department Committee on Graduate Students

Rings of Regular Functions on Spherical Nilpotent Orbit in Classical Groups

by
Tonghoon Suk

Submitted to the Department of Mathematics
on August 7, 2009, in partial fulfillment of the
requirements for the degree of
Master of Science in Mathematics

Abstract

Let G be a classical group and let $\mathfrak{g}$ be its Lie algebra. For a nilpotent element $X \in \mathfrak{g}$, the ring $R(\mathcal{O}_X)$ of regular functions on the nilpotent orbit $\mathcal{O}_X$ is a G-module. In this thesis, we will decompose it into irreducible representations of G for some spherical nilpotent orbits.

Thesis Supervisor: David Alexander Vogan
Title: Professor of Mathematics

1 Introduction

Let G be a classical complex group and let $\mathfrak{g}$ be its Lie algebra. An element $X \in \mathfrak{g}$ is called *nilpotent* if $X \in [\mathfrak{g}, \mathfrak{g}]$ and ad_X is a nilpotent endomorphism of the vector space $\mathfrak{g}$. For a nilpotent element $X \in \mathfrak{g}$, define the orbit through X by

$$\mathcal{O}_X = G_{ad} \cdot X = \{\mathrm{Ad}(g)X | g \in G\}.$$

For a nilpotent orbit in G, we can attach a partition by the Jordan normal form for the nilpotent as a matrix. [2]
A nilpotent orbit $\mathcal{O}_X$ is called *spherical* if a Borel subgroup of G has an open orbit in $\mathcal{O}_X$. All spherical nilpotent orbits of classical groups are introduced in Table 1.1. [1]

Table 1.1: Nilpotent orbits

G	partition of nilpotent orbit
$GL(n, \mathbb{C})$	$(2^k, 1^{n-2k})$ for $0 \leq k \leq \frac{n}{2}$
$Sp(2n, \mathbb{C})$	$(2^k, 1^{2n-2k})$ for $0 \leq k \leq n$
$SO(n, \mathbb{C})$	$(2^{2k}, 1^{n-4k})$ for $0 \leq k \leq \frac{n}{4}$
	$(3, 2^{2k}, 1^{n-4k-3})$ for $0 \leq k \leq \frac{n-3}{4}$

The ring of regular functions $R(\mathcal{O}_X)$ on a nilpotent orbit $\mathcal{O}_X$ of $\mathfrak{g}$ has a G-module structure and we will decompose it for all spherical orbits in $GL(n, \mathbb{C})$ and $Sp(2n, \mathbb{C})$. For the case of $SO(2n, \mathbb{C})$ and $SO(2n+1, \mathbb{C})$, we will do this for nilpotent orbits attached to the partitions involving only 1 and 2, or 1 and 3.

Since $\mathcal{O}_X = G/G^X$ where G^X is the stabilizer of X in G,

$$R(\mathcal{O}_X) = \mathrm{Ind}_{G^X}^{G}(1)$$

where 1 is the trivial representation. To decompose the representation $\mathrm{Ind}_{G^X}^{G}(1)$ for spherical orbits, we introduce a parabolic subgroup P of G containing G^X. Then we have

$$\mathrm{Ind}_{G^X}^{G}(1) = \mathrm{Ind}_{P}^{G} \mathrm{Ind}_{G^X}^{P}(1).$$

Using Helgason's Theorem and its corollary, we can easily decompose $\mathrm{Ind}_{G^X}^{P}(1)$.

Suppose that G is a complex connected reductive algebraic group and defined over $\mathbb{R}$; write $G(\mathbb{R})$ for the group of real points, which is a real reductive Lie group. We fix an Iwasawa decomposition $G(\mathbb{R}) = K(\mathbb{R}) \times A \times N(\mathbb{R})$ by Theorem 7.31 in Knapp [3]. Here $K(\mathbb{R})$ is a maximal compact subgroup of $G(\mathbb{R})$; this notation is justified because the complexification K of $K(\mathbb{R})$ is in fact an algebraic subgroup of G, defined over $\mathbb{R}$, with real points $K(\mathbb{R})$. Let $\mathfrak{g}_0$ be the Lie algebra of $G(\mathbb{R})$ and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ be the corresponding Iwasawa decomposition of $\mathfrak{g}$. Let $\mathfrak{t}_0$ be a maximal abelian subspace of $Z_{\mathfrak{k}_0}(\mathfrak{a}_0)$, so that $\mathfrak{t}_0 \oplus \mathfrak{a}_0$ is a maximally noncompact Cartan subalgebra of $\mathfrak{g}_0$. Finite

dimensional representations of $G(\mathbb{R})$ yield representation of $\mathfrak{g}_0$, hence complex-linear representations of $\mathfrak{g}$, which is the Lie algebra of G. Then we use the complexification of $\mathfrak{t}_0 \oplus \mathfrak{a}_0$ as Cartan subalgebra of $\mathfrak{g}$.
Let Δ and Σ be the sets of roots and restricted roots, respectively, and let Σ^+ be the set of positive restricted roots relative to $\mathfrak{n}_0$.
Roots and weights are real on $i\mathfrak{t}_0 \oplus \mathfrak{a}_0$, and we introduce an ordering such that nonzero restriction to $\mathfrak{a}_0$ of a member of Δ^+ is a member of Σ^+. A restricted weight of a finite-dimensional representation is the restriction to $\mathfrak{a}_0$ of a weight. Because of the choice of ordering, the restriction to $\mathfrak{a}_0$ of the highest weight of a finite-dimensional representation is the highest restricted weight. With this root system, we have Helgason's Theorem (Theorem 8.49 in Knapp [3]).

Theorem 1 (Helgason). *Suppose that G is a complex connected reductive algebraic group and defined over $\mathbb{R}$. Let $G(\mathbb{R})$ be the group of real points. Then, for an irreducible finite-dimensional representation π of $G(\mathbb{R})$, the following statements are equivalent:*
(a) π has a nonzero $K(\mathbb{R})$ fixed vector,
(b) $Z_{K(\mathbb{R})}(\mathfrak{a}_0)$ acts by the 1-dimensional trivial representation in the highest restricted-weight space of π.

Corollary 1. *Let U be a compact connected Lie group with Lie algebra $\mathfrak{u}$, let K be the set of fixed elements under an involution Φ. Then, $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{q}$ where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{q}$ is -1 eigenspace in $\mathfrak{u}$ of $\mathrm{d}\,\Phi$. Choose a maximal abelian subspace $\mathfrak{b}$ of $\mathfrak{q}$, let $\mathfrak{s}$ be a maximal abelian subspace of $Z_{\mathfrak{k}}(\mathfrak{b})$, and put $\mathfrak{t} = \mathfrak{b} \oplus \mathfrak{s}$. Impose an ordering on $(i\mathfrak{t})'$ that takes $i\mathfrak{b}$ before $i\mathfrak{s}$. Suppose that L is a subgroup which is contained in K and contains the identity component of K. Then an irreducible finite-dimensional representation π of U has a nonzero L fixed vector if and only if $Z_L(\mathfrak{b})$ fixes a nonzero highest-weight vector of ϕ.*

To apply Helgason's Theorem, we will use Theorem 7.1 in Procesi [5].

Theorem 2. *Suppose H is a complex reductive algebraic group defined over $\mathbb{R}$, and that $H(\mathbb{R})$ is a compact form of H. Then the restriction map, from the space of regular functions on H to the space of $H(\mathbb{R})$-finite functions on $H(\mathbb{R})$, is an isomorphism.*

After decomposing $\mathrm{Ind}_{G^X}^{P}(1)$, we will use Borel-Weil Theorem to decompose $\mathrm{Ind}_{P}^{G}\,\mathrm{Ind}_{G^X}^{P}(1)$.

2 General Linear Groups

Theorem 3. *Suppose $0 \le k \le \frac{n}{2}$, and that $\mathcal{O}_k$ is the nilpotent $GL(n,\mathbb{C})$-orbit attached to the partition $(2^k, 1^{n-2k})$. Then the ring $R(\mathcal{O}_k)$ of regular functions on $\mathcal{O}_k$ is isomorphic to the direct sum of all irreducible representations of $GL(n,\mathbb{C})$ whose highest weight is $(a_1, a_2, \cdots, a_k, 0, \cdots, 0, -a_k, -a_{k-1}, \cdots, -a_1)$ where $a_1 \ge a_2 \ge \cdots \ge a_k \ge 0$.*

Put $G = GL(n, \mathbb{C})$. For $0 \leq k \leq \frac{n}{2}$, define

$$X_k = \begin{pmatrix} 0_{k,n-k} & I_k \\ 0_{n-k,n-k} & 0_{n-k,k} \end{pmatrix}$$

where $0_{m,n}$ denotes the $m \times n$ zero matrix and I_k is the $k \times k$ identity matrix. Then the orbit $G \cdot X_k$ is the nilpotent orbit corresponding to the partition $(2^k, 1^{n-2k})$ of n according to Corollary 3.2.7 in Collingwood and McGovern [2].

Let G^{X_k} be the stabilizer of X_k in G. Then, an element $g \in G^{X_k}$ is of the form

$$g = \begin{pmatrix} A & * & * \\ 0_{n-2k,k} & B & * \\ 0_{k,k} & 0_{k,n-2k} & A \end{pmatrix}$$

where A is a $k \times k$ invertible matrix and B is a $(n-2k) \times (n-2k)$ invertible matrix.

Define a parabolic subgroup P_k whose elements are of the form

$$g = \begin{pmatrix} A & * & * \\ 0_{n-2k,k} & B & * \\ 0_{k,k} & 0_{k,n-2k} & C \end{pmatrix}$$

where A and C are $k \times k$ invertible matrices and B is a $(n-2k) \times (n-2k)$ invertible matrix. Then $P_k = GL(k, \mathbb{C}) \times GL(k, \mathbb{C}) \times GL(n-2k, \mathbb{C}) \rtimes R_U(P_k)$ where $R_U(P_k)$ is the unipotent radical of P_k and $R_U(P_k)$ centralize X_k. So, by Theorem 2,

$$\mathrm{Ind}_{G^{X_k}}^{P_k}(1) = \mathrm{Ind}_{\mathrm{diag}\, GL_(k,\mathbb{C})}^{GL(k,\mathbb{C}) \times GL(k,\mathbb{C})}(1) = \mathrm{Ind}_{\mathrm{diag}\, U(k)}^{U(k) \times U(k)}(1)$$

where $\mathrm{diag}\, U(k)$ is the diagonal copy of $U(k)$ in $U(k) \times U(k)$.

Define an involution Φ on $U(k) \times U(k)$ by $\Phi(X, Y) = (Y, X)$. Then $\mathrm{diag}\, U(k)$ is the set of fixed elements of Φ and we can consider this case from the point of view of Corollary 1. If $\mathfrak{c}$ is the subalgebra of diagonal matrices in $\mathfrak{u}(k)$, then we can take $\mathfrak{b} = \{(X, -X) | X \in \mathfrak{c}\}$. Let $\mathfrak{s} = \mathrm{diag}\, \mathfrak{c}$ and we have $\mathfrak{t} = \mathfrak{b} \oplus \mathfrak{c}$. On the other hand, roots in $U(k) \times U(k)$ are of the form $(\alpha, 0)$ and $(0, \alpha)$ with $\alpha \in \Delta_{U(k)}$ and their corresponding decompositions are $\frac{1}{2}(\alpha, -\alpha) + \frac{1}{2}(\alpha, \alpha)$ and $\frac{1}{2}(-\alpha, \alpha) + \frac{1}{2}(\alpha, \alpha)$. According to the hypotheses of Corollary 1, $(\alpha, 0) > 0$ implies $(0, -\alpha) > 0$. Consequently, the positive roots of $U(k) \times U(k)$ are

$$\Delta^+_{U(k) \times U(k)} = \{(\alpha, 0) | \alpha \in \Delta^+_{U(k)}\} \cup \{(0, -\alpha) | \alpha \in \Delta^+_{U(k)}\}.$$

Therefore, a weight (λ_1, λ_2) is dominant if λ_1 and $-\lambda_2$ are dominant for $\Delta^+_{U(k)}$. Since every irreducible representation of $U(k) \times U(k)$ is an outer tensor product of irreducible representations of $U(k)$, the irreducible representation of $U(k) \times U(k)$ with highest weight (λ_1, λ_2) is the outer tensor product $\tau_1 \otimes \tau_2$. Then τ_1 is equivalent to an irreducible representation with highest weight λ_1 and τ_2 has lowest weight λ_2.

The group $M = Z_{\mathrm{diag}\, U(k)}(\mathfrak{b})$ is the subgroup of elements (x, x) in $U(k) \times U(k)$ with $\mathrm{Ad}(x, x)(X, -X) = (X, -X)$ for all $X \in \mathfrak{c}$ and so, $M = \exp \mathfrak{s}$. The condition of Corollary 1 is that (λ_1, λ_2) vanish on $\mathfrak{s}$, hence that $\lambda_1 + \lambda_2 = 0$.

Therefore, $\mathrm{Ind}_{G^{X_k}}^{P_k}(1)$ is equivalent to the direct sum of $\tau_{(a_1, a_2, \cdots, a_k)} \otimes \tau_{(-a_k, -a_{k-1}, \cdots, -a_1)}$ where $a_1 \geq a_2 \geq \cdots \geq a_k$ are integers and $\tau_{(a_1, a_2, \cdots, a_k)}$ is an irreducible representation of $U(k)$ with highest weight $(a_1, a_2, \cdots, a_k)$.

For each summand $\tau_{(a_1,a_2,\cdots,a_k)}\otimes\tau_{(-a_k,-a_{k-1},\cdots,-a_1)}$, by Borel-Weil Theorem, can be expressed as

$$\mathrm{Ind}_{B(k)\times B(n-2k)\times B(k)\rtimes R_U(G)}^{GL(k,\mathbb{C})\times GL(n-2k,\mathbb{C})\times GL(k,\mathbb{C})\rtimes R_U(G)}(\xi_{(-a_1,-a_2,\cdots,-a_k)}\otimes 1\otimes\xi_{(a_k,a_{k-1},\cdots,a_1)})$$

where $B(k)$ is the Borel subgroup of upper triangular matrices of $GL(k,\mathbb{C})$, $\xi_{(a_1,a_2,\cdots,a_k)}$ is 1 dimensional representation corresponding to the weight $(a_1,a_2,\cdots,a_k)$ of $B(k)$ and $R_U(G)$ acts trivially. On the other hand, the subgroup $B(k)\times B(n-2k)\times B(k)\rtimes R_U(G)$ is the Borel subgroup $B(n)$ of upper triangular matrix of $GL(n,\mathbb{C})$ and the representation $\xi_{(-a_1,-a_2,\cdots,-a_k)}\otimes 1\otimes\xi_{(a_k,a_{k-1},\cdots,a_1)}$ is the 1 dimensional representation of $B(n)$ which corresponds the weight $(-a_1,-a_2,\cdots,-a_k,0,\cdots,0,a_k,a_{k-1},\cdots,a_1)$.
$\mathrm{Ind}_{B(n)}^{GL(n,\mathbb{C})}\xi_{(-a_1,-a_2,\cdots,-a_k,0,\cdots,0,a_k,a_{k-1},\cdots,a_1)}$ is an irreducible representation of $GL(n,\mathbb{C})$ with highest weight $(a_1,a_2,\cdots,a_k,0,\cdots,0,-a_k,-a_{k-1},\cdots,-a_1)$ if $a_1\geq\cdots\geq a_k\geq 0$ by Borel-Weil Theorem.
Therefore, $R(\mathcal{O}_k)$ is isomorphic to the direct sum of all irreducible representations of $GL(n,\mathbb{C})$ whose highest weight is of the form $(a_1,a_2,\cdots,a_k,0,\cdots,0,-a_k,-a_{k-1},\cdots,-a_1)$ where $a_1\geq a_2\geq\cdots\geq a_k\geq 0$.

3 Symplectic Groups

Theorem 4. *Suppose $0\leq k\leq n$, and that $\mathcal{O}_k$ is the nilpotent $Sp(2n,\mathbb{C})$-orbit attached to the partition $(2^k,1^{2n-2k})$ of $2n$. Then the ring $R(\mathcal{O}_k)$ of regular functions on $\mathcal{O}_k$ is isomorphic to the direct sum of all irreducible representations of $Sp(2n,\mathbb{C})$ whose highest weight is of the form $(a_1,a_2,\cdots,a_k,0,\cdots,0)$ where $a_1\geq a_2\geq\cdots\geq a_k\geq 0$ are even integers.*

Suppose $(\mathbb{C}^{2n},\omega)$ is a symplectic vector space with standard basis $\{e_1,\cdots,e_n,f_1,\cdots,f_n\}$, and let $G=Sp(2n,\mathbb{C})$. Then G acts on $\mathbb{C}^{2n}$. For $0\leq k\leq n$, define

$$X_k=\begin{pmatrix}0_{n,n} & I_{k,n}\\ 0_{n,n} & 0_{n,n}\end{pmatrix},$$

where $I_{k,n}$ denotes the $n\times n$ matrix $\begin{pmatrix}I_k & 0_{k,n-k}\\ 0_{n-k,k} & 0_{n-k,n-k}\end{pmatrix}$. Then G-orbit $\mathcal{O}_k$ containing X_k is the nilpotent orbit corresponding to the partition $(2^k,1^{2n-2k})$ according to Proposition 5.2.3 in Collingwood and McGovern [2].

Let G^{X_k} be the stabilizer of X_k in G. Using the following definition 1 and Lemma 1, we can find a parabolic subgroup containing G^{X_k}.

Definition 1. Suppose V is a complex vector space endowed with a symmetric or skew-symmetric bilinear form f. An *isotropic flag* in V is a chain of subspaces

$$\mathcal{W}=\{0=W_{r+1}\subset W_r\subset\cdots\subset W_1\}$$

so that the $f|_{W_1}=0$.

Let $G(V, f)$ be the group of automorphism of V preserving f.
Define $P(\mathcal{W}) = \{g \in G(V, f) | gW_i \subset W_i, 1 \leq i \leq r\}$. For $g \in P(\mathcal{W})$ and $1 \leq i \leq r$, define $l_i(g) \in GL(W_i/W_{i+1})$ to be the linear transformation induced by g.
For $i = 0$, define

$$V_0 = W_1^{\perp}/W_1,$$

which inherits a nondegenerate form f_0 from V. Since g preserves W_1 and f, it induces a linear transformation

$$l_0(g) \in G(V_0, f_0).$$

Lemma 1. *Suppose $\mathcal{W}$ is an isotropic flag in V. Then $P(\mathcal{W})$ is a parabolic subgroup of $G(f)$. Its unipotent radical is*

$$U(\mathcal{W}) = \{g \in P(\mathcal{W}) |\ l_i(g) \in GL(W_i/W_{i+1}) = \mathrm{id} \quad \textit{for } 1 \leq i \leq r,\ \ l_0(g) = \mathrm{id}\}$$

The maps l_i and l_0 identify the reductive quotient P/U with

$$GL(W_r) \times GL(W_{r-1}/W_r) \times \cdots \times GL(W_1/W_2) \times G(W_1^{\perp}/W_1, f_0).$$

Let W be the span of $\{e_1, e_2, \cdots, e_k\}$. and let P_k be the stabilizer of W. Since W is an isotropic subspace, P_k is a parabolic subgroup. Also, P_k contains G^{X_k} because $W = \mathrm{Im} X_k$. So,

$$R(\mathcal{O}_k) = \mathrm{Ind}_{G^{X_k}}^{G}(1) = \mathrm{Ind}_{P_k}^{G} \mathrm{Ind}_{G^{X_k}}^{P_k}(1)$$

where 1 is the trivial representation.

If we write the matrices in an ordered basis $\{e_1, e_2, \cdots, e_n, f_{k+1}, f_{k+2}, \cdots, f_n, f_1, f_2, \cdots, f_k\}$, we have

$$P_k = GL(k, \mathbb{C}) \times Sp(2n-2k, \mathbb{C}) \rtimes R_U(P_k),$$
$$G^{X_k} = O(k, \mathbb{C}) \times Sp(2n-2k, \mathbb{C}) \rtimes R_U(G^{X_k}).$$

But, by lemma 1, $R_U(P_k) = \{g \in Sp(2n, \mathbb{C}) |\ g|_W = \mathrm{id},\ g|_{W^{\perp}/W} = \mathrm{id}\}$. Then, it is easy to check that $g|_{\mathbb{C}^{2n}/W^{\perp}} = \mathrm{id}$ for $g \in R_U(P_k)$. Since $W^{\perp} = \mathrm{Ker}\, X_k$, $gX|_{W^{\perp}} = Xg|_{W^{\perp}} = 0$. For $1 \leq i \leq k$, $g(f_i) \in f_i + W^{\perp}$. So, $Xg(f_i) = gX(f_i) = e_i$. Therefore, $R_U(P_k)$ centralize X_k and

$$\mathrm{Ind}_{G^{X_k}}^{P_k}(1) = \mathrm{Ind}_{O(k,\mathbb{C})}^{GL(k,\mathbb{C})}(1) = \mathrm{Ind}_{O(k)}^{U(k)}(1)$$

where $Sp(2n-2k, \mathbb{C}) \rtimes R_U(P_k)$ acts trivially.

To apply Helgason's Theorem, define an involution Φ on $U(k)$ by $\Phi(X) = {}^tX^{-1}$. Then $O(k)$ is the set of fixed elements of Φ. Since $\mathfrak{q}$ in Corollary 1 is set of symmetric matrices in $\mathfrak{u}(n)$, we can take $\mathfrak{b}$ as the set of diagonal matrices in $\mathfrak{u}(n)$. So, the ordering on $(i\mathfrak{t})^*$ is same as the standard one. The group $M = Z_{O(k)}(\mathfrak{b})$ is the set of diagonal matrices whose entry is 1 or -1. So, $\mathrm{Ind}_{O(k)}^{U(k)}(1)$ is the direct sum of all irreducible representations of $U(k)$ with highest highest weight $(a_1, a_2, \cdots, a_k)$ where $a_1 \geq a_2 \geq \cdots \geq a_k$ are even integers.

By Borel-Weil Theorem, each irreducible summand with highest weight $(a_1, a_2, \cdots, a_k)$ can be written as

$$\mathrm{Ind}_{B_1 \times B_2 \rtimes R_U(P_k)}^{P_k}(\xi_{(-a_1,-a_2,\cdots,-a_k)})$$

where B_1 and B_2 are the standard Borel subgroups of $GL(k, \mathbb{C})$ and $Sp(2n-2k, \mathbb{C})$, respectively and, $\xi_{(-a_1,-a_2,\cdots,-a_k)}$ is 1 dimensional representation of B_1 corresponding to the weight $(-a_1, -a_2, \cdots, -a_k)$. If we use the ordered basis $\{e_1, e_2, \cdots, e_n, f_1, f_2, \cdots, f_n\}$, $B_1 \times B_2 \rtimes R_U(P_k)$ is the standard Borel subgroup B of $Sp(2n, \mathbb{C})$ and the representation $\xi_{(-a_1,a_2,\cdots,-a_k)}$ is the 1 dimensional representation of B which corresponds to the weight $(-a_1, -a_2, \cdots, -a_k, 0, \cdots, 0)$. So, $\mathrm{Ind}_B^{Sp(2n,\mathbb{C})}\, \xi_{(-a_1,a_2,\cdots,-a_k,0,\cdots,0)}$ is an irreducible representation of $Sp(2n, \mathbb{C})$ with highest weight $(a_1, a_2, \cdots, a_k, 0, \cdots, 0)$ when $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$.
As a consequence, $R(\mathcal{O}_k)$ is isomorphic to the direct sum of all all irreducible representations of $Sp(2n, \mathbb{C})$ whose weight is of the form $(a_1, a_2, \cdots, a_k, 0, \cdots, 0)$ where $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$ are even integers.

4 Special Orthogonal Groups

For $SO(2n, \mathbb{C})$, the rings of regular functions on some spherical nilpotent orbits are decomposed as following:

Theorem 5. *(i) For $0 \leq k < \frac{n}{2}$, let $\mathcal{O}_k$ be the nilpotent $SO(2n, \mathbb{C})$-orbit attached to the partition $(2^{2k}, 1^{2n-4k})$. Then the ring $R(\mathcal{O}_k)$ of regular functions on $\mathcal{O}_k$ is decomposed as the direct sum of all irreducible representations of $SO(2n, \mathbb{C})$ with highest weight of $(a_1, a_1, a_2, a_2, \cdots, a_k, a_k, 0, \cdots, 0)$ where $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$ are integers.*

(ii) For $k = \frac{n}{2}$ (n is a multiple of 2), there are two nilpotent $SO(2n, \mathbb{C})$-orbits $\mathcal{O}_I$ and $\mathcal{O}_{II}$ attached to the partition (2^{2k}). The ring $R(\mathcal{O}_I)$ of regular functions on $\mathcal{O}_I$ is the direct sum of all irreducible representations of $SO(2n, \mathbb{C})$ with highest weight $(a_1, a_1, a_2, a_2, \cdots, a_k, a_k)$ and the ring $R(\mathcal{O}_{II})$ of regular functions on $\mathcal{O}_{II}$ is decomposed as the direct sum of all irreducible representations of $SO(2n, \mathbb{C})$ with highest weight of $(a_1, a_1, a_2, a_2, \cdots, a_k, -a_k)$ where $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$ are integers.

(iii) *Let $\mathcal{O}$ be the nilpotent $SO(2n,\mathbb{C})$-orbit attached to the partition $(3,1^{2n-3})$. $R(\mathcal{O})$ is decomposed as the direct sum of all irreducible representations of $SO(2n,\mathbb{C})$ with highest weight of $(p,q,0,\cdots,0)$ where $p \geq |q|$ are integers and $p+q$ is even. For $n>2$, $q \geq 0$.*

Suppose $(\mathbb{C}^{2n},\omega)$ is a vector space with a non-degenerate symmetric bilinear form and let $G = SO(2n,\mathbb{C})$. Denote the standard basis $\{e_1,e_2,\cdots,e_n,f_1,f_2,\cdots,f_n\}$, so that $(e_i,e_j)=(f_i,f_j)=0$ and $(e_i,f_j)=\delta_{ij}$.

For $0 \leq k \leq \frac{n}{2}$, put

$$X_k = \begin{pmatrix} 0_{n,n} & I_{2k,n} \\ 0_{n,n} & 0_{n,n} \end{pmatrix}$$

where $I_{2k,n}$ denotes the $n \times n$ matrix $\begin{pmatrix} 0_{k,k} & I_k & 0_{k,n-2k} \\ -I_k & 0_{k,k} & 0_{k,n-2k} \\ 0_{n-2k,k} & 0_{n-2k,k} & 0_{n-2k,n-2k} \end{pmatrix}$.

Then $\mathcal{O}_k = G \cdot X_k$ is the nilpotent orbit corresponding to the partition $(2^{2k},1^{2n-4k})$ of $2n$ according to Corollary 5.2.8 in Collingwood and McGovern [2]. (If $k=\frac{n}{2}$, $G\cdot X_k$ is $\mathcal{O}_I$.)

Let W be the span of $\{e_1,e_2,\cdots,e_{2k}\}$ and P_k be the stabilizer of W. Since W is isotropic, P_k is a parabolic subgroup. Also, the stabilizer G^{X_k} of X_k is contained in P_k because $W = \mathrm{Im} X_k$.

If we write the matrices in an ordered basis $\{e_1,e_2,\cdots,e_n,f_{2k+1},f_{2k+2},\cdots,f_n,f_1,f_2,\cdots,f_{2k}\}$, we have

$$P_k = GL(2k,\mathbb{C}) \times SO(2n-4k,\mathbb{C}) \rtimes R_U(P_k),$$
$$G^{X_k} = Sp(2k,\mathbb{C}) \times SO(2n-4k,\mathbb{C}) \rtimes R_U(G^{X_k}).$$

$R_U(P_k)$ centralize X_k by the same reason in symplectic case. So,

$$\mathrm{Ind}_{G^{X_k}}^{P_k}(1) = \mathrm{Ind}_{Sp(2k,\mathbb{C})}^{GL(2k,\mathbb{C})}(1) = \mathrm{Ind}_{Sp(2k)}^{U(2k)}(1)$$

where $SO(2n-4k,\mathbb{C}) \rtimes R_U(P_k)$ acts trivially. $\mathrm{Ind}_{Sp(2k)}^{U(2k)}(1)$ is the direct sum of irreducible representations of $U(2k)$ that have $Sp(2k)$-fixed vector.

Lemma 2. *$\mathrm{Ind}_{Sp(2k)}^{U(2k)}(1)$ is the direct sum of all irreducible representations of $U(2k)$ with highest weight $(a_1,a_1,a_2,a_2,\cdots,a_k,a_k)$ where $a_1 \geq a_2 \geq \cdots \geq a_k$ are integers.*

Proof. $Sp(2k)$ is the set of fixed elements of an involution Φ on $U(2k)$ such that $\Phi(g) = J^t g^{-1} J^{-1}$ where J is the $2k \times 2k$ matrix $\begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$.

Let $\mathfrak{u}(2k) = \mathfrak{k} \oplus \mathfrak{q}$ be eigenspace decomposition under $d\Phi$ and $\mathfrak{s}$ and $\mathfrak{b}$ be the subalgebra of all diagonal matrices in $\mathfrak{k}$ and $\mathfrak{q}$. Then, $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{u}(2k)$ and a member $(a_1, a_2, \cdots, a_{2k})$ in $(i\mathfrak{t})^*$ decomposes as ;

$$\begin{aligned}(a_1, a_2, \cdots, a_{2k}) = &(\frac{a_1 + a_{k+1}}{2}, \cdots, \frac{a_k + a_{2k}}{2}, \frac{a_1 + a_{k+1}}{2}, \cdots, \frac{a_k + a_{2k}}{2}) \\ &+ (\frac{a_1 - a_{k+1}}{2}, \cdots, \frac{a_k - a_{2k}}{2}, -\frac{a_1 - a_{k+1}}{2}, \cdots, -\frac{a_k - a_{2k}}{2})\end{aligned}$$

with the first term carried on $i\mathfrak{b}$ and the second term carried on $i\mathfrak{s}$. According to the hypotheses of the Corollary 1, a root $e_i - e_j$ of $\mathfrak{u}(2k)$ is positive if $0 < i < j \leq k$ or $k < i < j \leq 2k$ or, $k < j$ and $i \leq j - k$. So, a weight $(a_1, a_2, \cdots, a_{2k})$ is dominant if $a_1 \geq a_2 \geq \cdots \geq a_k$, $a_{k+1} \geq a_{k+2} \geq \cdots \geq a_{2k}$, $a_i \geq a_{i+k}$ for $i = 1, 2, \cdots, k$ and $a_{k+i} \geq a_{i+1}$ for $i = 1, 2, \cdots, k-1$. Since the group $M = Z_{Sp(2k)}(\mathfrak{b})$ is compact and connected, M acts by 1-dimensional trivial representation on the highest weight space if the highest weight $(a_1, a_2, \cdots, a_{2k})$ is zero on $\mathfrak{s}$. So, $a_i = a_{k+i}$ for $i = 1, 2, \cdots, k$. Therefore, with the standard positive root system, $\mathrm{Ind}_{Sp(2k)}^{U(2k)}(1)$ is the direct sum of all irreducible representation of $U(2k)$ with highest weight $(a_1, a_1, a_2, a_2, \cdots, a_k, a_k)$ where $a_1 \geq a_2 \geq \cdots \geq a_k$. □

By Borel-Weil Theorem, each summand of irreducible representation with highest weight $(a_1, a_2, \cdots, a_{2k})$ can be written as

$$\mathrm{Ind}_{B_1 \times B_2 \rtimes R_U(P_k)}^{P_k}(\xi_{(-a_1, -a_2, \cdots, -a_{2k})})$$

where B_1 is the Borel subgroup of upper triangular matrices in $GL(k, \mathbb{C})$, B_2 are the standard Borel subgroup of $SO(2n - 4k, \mathbb{C})$ and $\xi_{(-a_1, -a_2, \cdots, -a_{2k})}$ is 1 dimensional representation corresponding to the weight $(-a_1, -a_2, \cdots, -a_{2k})$. Using the ordered basis $\{e_1, e_2, \cdots, e_n, f_1, f_2, \cdots, f_n\}$, $B_1 \times B_2 \rtimes R_U(P_k)$ becomes the standard Borel subgroup B of $SO(2n, \mathbb{C})$ and the representation $\xi_{(-a_1, -a_2, \cdots, -a_{2k})}$ is the 1 dimensional representation of B which corresponds the weight $(-a_1, \cdots, -a_{2k}, 0, \cdots, 0)$. By Borel-Weil Theorem, $\mathrm{Ind}_B^{SO(2n, \mathbb{C})} \xi_{(-a_1, -a_2, \cdots, -a_{2k}, 0, \cdots, 0)}$ is an irreducible representation of $SO(2n, \mathbb{C})$ with highest weight $(a_1, a_2, \cdots, a_{2k}, 0, \cdots, 0)$ when $a_1 \geq a_2 \geq \cdots \geq a_{2k} \geq 0$.
Therefore, $R(\mathcal{O}_{2k})$ is isomorphic to the direct sum of all all irreducible representations of $SO(2n, \mathbb{C})$ whose weight is of the form $(a_1, a_1, a_2, a_2 \cdots, a_k, a_k, 0, \cdots, 0)$ where $a_1 \geq a_2 \cdots \geq a_k \geq 0$ are integers.

For $n = 2k$, there is another nilpotent orbit $\mathcal{O}_{II}$ with partition (2^{2k}) by Theorem 5.1.4 in Collingwood and McGovern [2]. If we interpret the matrix X_k defined above as an element of $SO(2n)$ using the ordered basis $\{e_1, e_2, \cdots, e_{n-1}, f_n, f_1, f_2, \cdots, f_{n-1}, e_n\}$, $G \cdot X_k$ is $\mathcal{O}_{II}$. So, $\mathrm{Ind}_{G^{X_k}}^{P_k}(1) = \mathrm{Ind}_{Sp(2k)}^{U(2k)}(1)$ is the direct sum of all irreducible representation of $U(2k)$ with highest weight $(a_1, a_1, a_2, a_2 \cdots, a_k, a_k)$ where $a_1 \geq a_2 \cdots \geq a_k$. Applying Borel-Weil Theorem and changing the ordered basis to $\{e_1, e_2, \cdots, e_n, f_1, f_2, \cdots, f_n\}$, $R(\mathcal{O}_{2k})$ is isomorphic to the direct sum of all irreducible representations of $SO(2n, \mathbb{C})$ with highest weight $(a_1, a_1, a_2, a_2 \cdots, a_{k-1}, a_{k-1}, a_k, -a_k)$ where $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$ are integers.

For the partition $(3, 1^{2n-3})$, define

$$X = E_{2,1} - E_{n+1,n+2} + E_{1,n+2} - E_{2,n+1}$$

where $E_{i,j}$ denotes the $2n \times 2n$ matrix whose i,j-entry is 1 and other entries are 0. Then $\mathcal{O} = G \cdot X$ is the nilpotent orbit corresponding to the partition by Proposition 5.2.8 in Collingwood and McGovern [2].

Let W be the span of $\{e_2\}$ and define P as the stabilizer of W. Since W is an isotropic subspaces, P is a parabolic subgroup. Also, the stabilizer G^X of X is contained in P because $W = \mathrm{Im} X^2$.

With the ordered basis $\{e_2, e_1, e_3, \cdots, e_n, f_1, f_3, \cdots, f_n, f_2\}$, we have

$$\begin{aligned} P &= GL(W) \times SO(W^\perp/W) \rtimes R_U(P), \\ G^X &= (GL(W) \times SO(W^\perp/W))^X \rtimes R_U(G^X) \end{aligned}$$

If $g \in R_U(P)$, by Lemma 1, $g|_{\mathrm{Im}X^2} = \mathrm{id}$ and $g|_{W^\perp/W} = \mathrm{id}$. Since $W \subset \mathrm{Ker}\, X$, $gX|_{W^\perp} = Xg|_{W^\perp}$. Let $g(e_1) = e_1 + ae_2$ and $g(f_1) = f_1 + be_2$. Since $g \in SO(2n, \mathbb{C})$, $g(f_2) = f_2 - af_1 - be_1 + v$ for some $v \in \mathrm{Ker}\, X$. So, $Xg(f_2) = gX(f_2) = e_1 - f_1 + (a-b)e_2$. Therefore, $R_U(P)$ is contained in G^X.

Lemma 3. $\mathrm{Ind}_{(GL(W)\times SO(W^\perp/W))^X}^{GL(W)\times SO(W^\perp/W)}(1)$ *is the direct sum of all irreducible representations of* $GL(W) \times SO(W^\perp/W)$ *with highest weight* $(p, q, 0, \cdots, 0)$ *where* $p + q$ *is even. If* $n > 2$, $q \geq 0$.

Proof. Put $G = GL(W) \times SO(W^\perp/W)$ and let $\mathfrak{g}$ be its Lie algebra. Also, put $L = G^X$.
Let $\overline{e_i} = e_i + W$ and $\overline{f_i} = f_i + W$ for $i = 1, 3, \cdots, n$.
Then $\{e_2, \frac{1}{\sqrt{2}}(\overline{e_1} - \overline{f_1}), \overline{e_3}, \cdots, \overline{e_n}, \overline{f_3}, \cdots, \overline{f_n}, \frac{1}{\sqrt{2}}(\overline{e_1} + \overline{f_1}), f_2\}$ is an ordered basis of G. With this ordered basis, X becomes $E = \sqrt{2}E_{1,2} + \sqrt{2}E_{2,2n}$ and element $g \in K$ is of the form

$$g = \begin{pmatrix} \det A & 0 & 0_{1,2n-3} & 0 \\ 0 & \det A & 0_{1,2n-3} & 0 \\ 0_{2n-3,1} & 0_{2n-3,1} & A & 0_{2n-3,1} \\ 0 & 0 & 0_{1,2n-3} & \det A \end{pmatrix}$$

where $A \in O(V)$ and V is the subspace spanned by $\{\overline{e_3}, \cdots, \overline{e_n}, \overline{f_3}, \cdots, \overline{f_n}, \frac{1}{\sqrt{2}}(\overline{e_1} + \overline{f_1})\}$.
Let $Y = E_{1,2n} + E_{2n} + I_{2n,2n} - E_{1,1} - 2E_{2,2}$ and Φ be the conjugation by Y. Let K be the set of all fixed elements of Φ. An element $g \in K$ is of the form

$$g = \begin{pmatrix} a & 0 & 0_{1,2n-3} & 0 \\ 0 & \det A & 0_{1,2n-3} & 0 \\ 0_{2n-3,1} & 0_{2n-3,1} & A & 0_{2n-3,1} \\ 0 & 0 & 0_{1,2n-3} & a \end{pmatrix}$$

where $a = \pm 1$ and $A \in O(V)$. Since the identity component of K is contained in L and K contains L, we can apply Corollary 1.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ be eigenspace decomposition under $d\,\Phi$. Define $\mathfrak{b}$ is subspace of $\mathfrak{q}$ spanned by $\{E_{1,1}, E_{2,2n-1} + E_{2n-1,2}\}$. Then, $\mathfrak{b}$ is a maximal abelian subalgebra of $\mathfrak{q}$ and an element $g \in M = Z_L(\mathfrak{b})$ is of the from

$$g = \begin{pmatrix} a & 0 & 0_{1,2n-4} & 0 & 0 \\ 0 & a & 0_{1,2n-4} & 0 & 0 \\ 0_{2n-4,1} & 0_{2n-4,1} & A & 0_{2n-4,1} & 0_{2n-4,1} \\ 0 & 0 & 0_{1,2n-4} & a & 0 \\ 0 & 0 & 0_{1,2n-4} & 0 & a \end{pmatrix}$$

where $a = \pm 1$ and $A \in SO(V_1)$ where V_1 is the subspace spanned by $\{\overline{e_3}, \cdots, \overline{e_n}, \overline{f_3}, \cdots, \overline{f_n}\}$. So, by Corollary 1, $\mathrm{Ind}_{G^x}^{G}(1)$ is the direct sum of all irreducible representation of G with highest weight $(p, q, 0, \cdots, 0)$ where $p + q$ is even. For $n > 2$, $q \geq 0$. $\square$

If we use the ordered basis $\{e_2, e_1, e_3, \cdots, e_n, f_1, f_3, \cdots, f_n, f_2\}$ and apply Borel-Weil Theorem, each summand of irreducible representation with highest weight $(p, q, 0, \cdots, 0)$ can be written as

$$\mathrm{Ind}_{GL(1,\mathbb{C}) \times B_1 \rtimes R_U(P)}^{P}(\xi_{(-p,-q,0,\cdots,0)})$$

where B_1 is the standard Borel subgroup of $SO(2n-2, \mathbb{C})$, and $\xi_{(-p,-q,0,\cdots,0)}$ is 1 dimensional representation corresponding to the weight $(-p, -q, 0, \cdots, 0)$. But,$GL(1, \mathbb{C}) \times B_1 \rtimes R_U(P)$ is the standard Borel subgroup B of $SO(2n, \mathbb{C})$ and $\xi_{(-p,-q,0,\cdots,0)}$ is the 1 dimensional representation of B which corresponds to the weight $(-p, -q, 0, \cdots, 0)$. Applying Borel-Weil Theorem again, $\mathrm{Ind}_B^{SO(2n,\mathbb{C})}\, \xi_{(-p,-q,0,\cdots,0)}$ is an irreducible representation of $SO(2n, \mathbb{C})$ with highest weight $(p, q, 0, \cdots, 0)$ when $p \geq |q|$. If $n > 2$, $p \geq 0$. Therefore, $R(\mathcal{O})$ is the direct sum of all irreducible representations of $SO(2n, \mathbb{C})$ with highest weight of $(p, q, 0, \cdots, 0)$ where $p \geq |q|$ are integers and $p + q$ is even. Also, for $n > 2$, $q \geq 0$.

For $SO(2n+1, \mathbb{C})$ the rings of regular functions on some spherical nilpotent orbits are decomposed as following:

Theorem 6. *(i) For $0 \leq k \leq \frac{2n+1}{4}$, let $\mathcal{O}_k$ be the nilpotent $SO(2n+1, \mathbb{C})$-orbit attached to the partition $(2^{2k}, 1^{2n+1-4k})$.
Then the ring $R(\mathcal{O}_k)$ of regular functions on $\mathcal{O}_k$ is decomposed as the direct sum of all irreducible representations of $SO(2n+1, \mathbb{C})$ with highest weight of $(a_1, a_1, a_2, a_2, \cdots, a_k, a_k, 0, \cdots, 0)$ where $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$.*

(ii) Let $\mathcal{O}$ be the nilpotent $SO(2n+1, \mathbb{C})$-orbit attached to the partition $(3, 1^{2n-2})$ with $n \geq 2$. Then the ring $R(\mathcal{O}_k)$ of regular functions on $\mathcal{O}_k$ is decomposed as the direct sum of all irreducible representations of $SO(2n+1, \mathbb{C})$ with highest weight of $(p, q, 0, \cdots, 0)$ where $p \geq q \geq 0$ are integers and $p + q$ is even.

Suppose $(\mathbb{C}^{2n+1}, \omega)$ is a vector space with a non-degenerate symmetric bilinear form and let $G = SO(2n+1, \mathbb{C})$. Denote the standard basis $\{e, e_1, e_2, \cdots, e_n, f_1, f_2, \cdots, f_n\}$, so that $(e, e) = 1, (e, e_i) = (e, f_i) = (e_i, e_j) = (f_i, f_j) = 0$ and $(e_i, f_j) = \delta_{ij}$.

For $0 \leq k \leq \frac{n}{4}$, put

$$X_k = \begin{pmatrix} 0_{1,1} & 0_{1,n} & 0_{1,n} \\ 0_{n,1} & 0_{n,n} & I_{k,n} \\ 0_{n,1} & 0_{n,n} & 0_{n,n} \end{pmatrix}$$

where $I_{k,n}$ denotes the $n \times n$ matrix $\begin{pmatrix} 0_{k,k} & I_k & 0_{k,n-2k} \\ -I_k & 0_{k,k} & 0_{k,n-2k} \\ 0_{n-2k,k} & 0_{n-2k,k} & 0_{n-2k,n-2k} \end{pmatrix}$.
Then $\mathcal{O}_k = G \cdot X_k$ is the nilpotent orbit corresponding to the partition $(2^{2k}, 1^{2n+1-4k})$ of $2n+1$ by Proposition 5.2.5 in Collingwood and McGovern [2].
If W is the span of $\{e_1, e_2, \cdots, e_{2k}\}$, then the stabilizer G^{X_k} of X_k stabilize the space W. So, the stabilizer of W, denoted by P_k, contains G^{X_k}. If we write the matrices in an ordered basis $\{e, e_1, e_2, \cdots, e_n, f_{2k+1}, f_{2k+2}, \cdots, f_n, f_1, f_2, \cdots, f_{2k}\}$, we have

$$P_k = GL(2k, \mathbb{C}) \times SO(2n-4k+1, \mathbb{C}) \rtimes R_U(P_k),$$
$$G^{X_k} = Sp(2k, \mathbb{C}) \times SO(2n-4k+1, \mathbb{C}) \rtimes R_U(G^{X_k}).$$

$R_U(P_k)$ is contained in G^{X_k} because of the same reason in $SO(2n, \mathbb{C})$ case. So, by Lemma 1, $\mathrm{Ind}_{G^{X_k}}^{P_k}(1)$ is the direct sum of all irreducible representation of $GL(2k, \mathbb{C})$ with highest weight $(a_1, a_2, \cdots, a_{2k})$ where $a_1 = a_2 \geq a_3 = a_4 \geq \cdots \geq a_{2k-1} = a_{2k}$.
By Borel-Weil Theorem, each summand of irreducible representation with highest weight $(a_1, a_2, \cdots, a_{2k})$ can be written as

$$\mathrm{Ind}_{B_1 \times B_2 \rtimes R_U(P_k)}^{P_k}(\xi_{(-a_1, -a_2, \cdots, -a_{2k})})$$

where B_1 is the Borel subgroup of upper triangular matrices in $GL(2k, \mathbb{C})$, B_2 are the standard Borel subgroup of $SO(2n-4k+1, \mathbb{C})$ and $\xi_{(-a_1, -a_2, \cdots, -a_{2k})}$ is 1 dimensional representation corresponding to the weight $(-a_1, -a_2, \cdots, -a_{2k})$. Using the ordered basis $\{e, e_1, e_2, \cdots, e_n, f_1, f_2, \cdots, f_n\}$, $B_1 \times B_2 \rtimes R_U(P_k)$ becomes the standard Borel subgroup B of $SO(2n+1, \mathbb{C})$ and the representation $\xi_{(-a_1, -a_2, \cdots, -a_{2k})}$ is the 1 dimensional representation of B which corresponds the weight $(-a_1, \cdots, -a_{2k}, 0, \cdots, 0)$. If we apply Borel-Weil Theorem, $\mathrm{Ind}_B^{SO(2n+1, \mathbb{C})} \xi_{(-a_1, -a_2, \cdots, -a_{2k}, 0, \cdots, 0)}$ is an irreducible representation of $SO(2n+1, \mathbb{C})$ with highest weight $(a_1, a_2, \cdots, a_{2k}, 0, \cdots, 0)$ when $a_1 \geq a_2 \geq \cdots \geq a_{2k} \geq 0$.
Therefore, $R(\mathcal{O}_k)$ is isomorphic to the direct sum of all all irreducible representations of $SO(2n+1, \mathbb{C})$ whose highest weight is of the form $(a_1, a_2, \cdots, a_{2k}, 0, \cdots, 0)$ where $a_1 = a_2 \geq a_3 = a_4 \geq \cdots \geq a_{2k-1} = a_{2k} \geq 0$ are integers.

For the partition $(3, 1^{2n-2})$ of $2n+1$, define

$$X = E_{3,2} - E_{n+2,n+3} + E_{2,n+3} - E_{3,n+2}$$

where $E_{i,j}$ is defined in $SO(2n, \mathbb{C})$ case. Then $\mathcal{O} = G \cdot X$ is the nilpotent orbit corresponding to the partition $(3, 1^{2n-2})$ of $2n+1$.
Let W be the span of $\{e_2\}$ and define P as the stabilizer of W. By the same reason for $SO(2n, \mathbb{C})$ case, P is a parabolic subgroup containing G^X.

Using the ordered basis $\{e_2, e, e_1, e_3, \cdots, e_n, f_1, f_3, \cdots, f_n, f_2\}$, we have

$$P = GL(W) \times SO(W^{\perp}/W) \rtimes R_U(P),$$
$$G^X = (GL(W) \times SO(W^{\perp}/W))^X \rtimes R_U(G^X).$$

$R_U(P)$ is contained in G^X by the same reason for $SO(2n, \mathbb{C})$ case.

Lemma 4. $\mathrm{Ind}_{(GL(W)\times SO(W^{\perp}/W))^X}^{GL(W)\times SO(W^{\perp}/W)}(1)$ *is the direct sum of all irreducible representations of* $GL(W) \times SO(W^{\perp}/W)$ *with highest weight* $(p, q, 0, \cdots, 0)$ *where* $p+q$ *is even and* $q \geq 0$.

Proof. Put $G = GL(W) \times SO(W^{\perp}/W)$ and let $\mathfrak{g}$ be its Lie algebra. Also, put $L = G^X$.
Let $\overline{e_i} = e_i + W$, $\overline{f_i} = f_i + W$ for $i = 1, 3, \cdots, n$ and $\overline{e} = e + W$.
Then $\{e_2, \frac{1}{\sqrt{2}}(\overline{e_1} - \overline{f_1}), \overline{e}, \overline{e_3}, \cdots, \overline{e_n}, \overline{f_3}, \cdots, \overline{f_n}, \frac{1}{\sqrt{2}}(\overline{e_1} + \overline{f_1}), f_2\}$ is an ordered basis of G.
With this ordered basis, X becomes $E = \sqrt{2}E_{1,2} + \sqrt{2}E_{2,2n+1}$ and element $g \in K$ is of the form

$$g = \begin{pmatrix} \det A & 0 & 0_{1,2n-2} & 0 \\ 0 & \det A & 0_{1,2n-2} & 0 \\ 0_{2n-2,1} & 0_{2n-2,1} & A & 0_{2n-2,1} \\ 0 & 0 & 0_{1,2n-2} & \det A \end{pmatrix}$$

where $A \in O(V)$ and V is the subspace spanned by $\{\overline{e}, \overline{e_3}, \cdots, \overline{e_n}, \overline{f_3}, \cdots, \overline{f_n}, \frac{1}{\sqrt{2}}(\overline{e_1} + \overline{f_1})\}$.
Let $Y = E_{1,2n+1} + E_{2n+1} + I_{2n+1,2n+1} - E_{1,1} - 2E_{2,2}$ and Φ be the conjugation by Y.
Let K be the set of all fixed elements of Φ. An element $g \in K$ is of the form

$$g = \begin{pmatrix} a & 0 & 0_{1,2n-2} & 0 \\ 0 & \det A & 0_{1,2n-2} & 0 \\ 0_{2n-2,1} & 0_{2n-2,1} & A & 0_{2n-2,1} \\ 0 & 0 & 0_{1,2n-2} & a \end{pmatrix}$$

where $a = \pm 1$ and $A \in O(V)$. Since the identity component of K is contained in L and K contains L, we can apply Corollary 1.
Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ be eigenspace decomposition under $\mathrm{d}\,\Phi$. Define $\mathfrak{b}$ is subspace of $\mathfrak{q}$ spanned by $\{E_{1,1}, E_{2,2n} + E_{2n,2}\}$. Then, $\mathfrak{b}$ is a maximal abelian subalgebra of $\mathfrak{q}$ and an element $g \in M = Z_L(\mathfrak{b})$ is of the from

$$g = \begin{pmatrix} a & 0 & 0_{1,2n-3} & 0 & 0 \\ 0 & a & 0_{1,2n-3} & 0 & 0 \\ 0_{2n-3,1} & 0_{2n-3,1} & A & 0_{2n-3,1} & 0_{2n-3,1} \\ 0 & 0 & 0_{1,2n-3} & a & 0 \\ 0 & 0 & 0_{1,2n-3} & 0 & a \end{pmatrix}$$

where $a = \pm 1$ and $A \in SO(V_1)$ where V_1 is the subspace spanned by $\{\overline{e}, \overline{e_3}, \cdots, \overline{e_n}, \overline{f_3}, \cdots, \overline{f_n}\}$.
So, by Corollary 1, $\mathrm{Ind}_{G^X}^{G}(1)$ is the direct sum of all irreducible representation of G with highest weight $(p, q, 0, \cdots, 0)$ where $p + q$ is even and $q \geq 0$. □

If we use the ordered basis $\{e, e_2, e_1, e_3, \cdots, e_n, f_1, f_3, \cdots, f_n, f_2\}$ and apply Borel-Weil Theorem, each summand of irreducible representation with highest weight $(p, q, 0, \cdots, 0)$ can be written as

$$\mathrm{Ind}_{GL(1,\mathbb{C})\times B_1 \rtimes R_U(P)}^{P}(\xi_{(-p,-q,0,\cdots,0)})$$

where B_1 is the standard Borel subgroup of $SO(2n-2, \mathbb{C})$ and $\xi_{(-p,-q,0,\cdots,0)}$ is 1 dimensional representation corresponding to the weight $(-p, -q, 0, \cdots, 0)$. But,$GL(1, \mathbb{C}) \times B_1 \rtimes R_U(P)$ is the standard Borel subgroup B of $SO(2n+1, \mathbb{C})$ and $\xi_{(-p,-q,0,\cdots,0)}$ is the 1 dimensional representation of B which corresponds to the weight $(-p, -q, 0, \cdots, 0)$. Applying Borel-Weil Theorem, $\mathrm{Ind}_B^{SO(2n+1,\mathbb{C})} \xi_{(-p,-q,0,\cdots,0)}$ is an irreducible representation of $SO(2n+1, \mathbb{C})$ with highest weight $(p, q, 0, \cdots, 0)$ when $p \geq q \geq 0$.
Therefore, $R(\mathcal{O})$ is the direct sum of all irreducible representations of $SO(2n+1, \mathbb{C})$ with highest weight of $(p, q, 0, \cdots, 0)$ where $p \geq q \geq 0$ are integers and $p+q$ is even.

References

[1] D. Panyushev, *Some amazing properties of spherical nilpotent orbits*, Mathematische Zeitschrift 245, 557-580 (2003)

[2] D. Collingwood and W. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold, New York

[3] A. W. Knapp, *Lie groups beyond an introduction*, Birhäuser, Boston, 1996, Second edition, 2002

[4] A. W. Knapp, *Representation theory of semisimple gorups*, Princeton university press, Princeton, 1986

[5] C. Procesi, *Lie Groups: An Approach through Invariants and Representations*, Springer, 2007

[6] W. McGovern, *Rings of regular functions on nilpotent orbits and their covers*, Invent. Math. 97 (1989), 209-217. MR 90g:22022