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Non-Abelian Spin Liquid in a Spin-One Quantum Magnet

Tarun Grover

Department of Physics, UC Berkeley, Berkeley, California 94720, USA

T. Senthil

Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

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We study a time-reversal invariant non-Abelian spin-liquid state in an $SU(2)$ symmetric spin $S = 1$ quantum magnet on a triangular lattice. The spin liquid is obtained by quantum disordering a noncollinear nematic state. We show that such a spin liquid cannot be obtained by the standard projective construction for spin liquids. We also study the phase transition between the spin liquid and the noncollinear nematic state and show that it cannot be described within the Landau-Ginzburg-Wilson paradigm.

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The concepts of “order parameter” and “symmetry breaking” have been extremely successful for classifying various phases of matter and their phase transitions at finite temperature. Interestingly, the past 20 years have provided us many examples where these notions break down in the context of zero temperature quantum phases and phase transitions and where one needs a totally different framework. In particular, concepts such as “fractionalization” and “topological order” have been shown to be very useful to describe the rich physics of various quantum Hall systems, frustrated magnets, and many other strongly correlated systems [1].

Our current description of topological order draws heavily on the projective construction using various slave boson or fermion techniques [1]. In this Letter, we present an example of a non-Abelian quantum spin liquid with bosonic spinons where the Schwinger boson construction fails to capture even the mean-field state. The spin liquid is obtained by quantum disordering a noncollinear nematic state in a spin $S = 1$ system on a triangular lattice. Interestingly, the low-energy excitations in the spin liquid have non-Abelian statistics. To our knowledge, this is one of the few known examples of a time-reversal invariant phase of matter that supports non-Abelian excitations [2,3]. Furthermore, the phase transition between the nematic phase and the spin liquid is also exotic and has a very large value of the critical exponent η associated with correlations of the nematic order parameter.

A general Hamiltonian describing a spin $S = 1$ quantum magnet on an isotropic triangular lattice takes the form

$$H = J \sum_{\langle ij \rangle} S_i \cdot S_j + K \sum_{\langle ij \rangle} (S_i \cdot S_j)^2. \quad (1)$$

Additionally, H may have other short-ranged interactions consistent with $SU(2)$ spin symmetry such as multiple ring exchange or second-nearest-neighbor Heisenberg exchange.

Consider the Hamiltonian H for $K > J > 0$. At the mean-field level, the ground state of this Hamiltonian in this parameter regime has a three-sublattice nematic order where the nematic directors on the three sublattices A , B , and C of the triangular lattice are orthogonal to each other (say, along \hat{x} , \hat{y} , and \hat{z} , respectively) [4,5]. As argued by Tsunetsugu and Arikawa [6], such a state may explain many of the features [7] of the triangular lattice magnet NiGa_2S_4 , in particular, the lack of any dipole moment $\langle S \rangle$, low temperature specific heat $C(T) \sim T^2$, and finite spin susceptibility at $T = 0$. The directors of the nematic correspond to “hard axes”; i.e., the spins on the three sublattices fluctuate in the plane perpendicular to their respective directors such that the average value $\langle S(\mathbf{r}) \rangle = 0$. Such a state breaks spin-rotation symmetry while preserving the time-reversal invariance. In terms of the spin operators, the nematic order parameter at site \mathbf{r} could be described as a rank-two tensor: $Q_{\mu\nu}(\mathbf{r}) = \frac{1}{2} \langle S_\mu(\mathbf{r}) S_\nu(\mathbf{r}) + S_\nu(\mathbf{r}) S_\mu(\mathbf{r}) \rangle - \frac{2}{3} \delta_{\mu\nu}$. The director at a site \mathbf{r} is along the eigenvector of $Q_{\mu\nu}$ that corresponds to the zero eigenvalue.

The above ground state does not preserve any continuous subgroup of the original $SO(3)$ symmetry of the Hamiltonian in Eq. (1). Thus the low-energy fluctuations around the ground state consist of three Goldstone modes. Interestingly, the ground state is invariant under the discrete subgroup $D_2 \equiv R_\pi^x, R_\pi^y, R_\pi^z$ of $SO(3)$, where R_π^a ($a = x, y, z$) corresponds to a global π rotation of all spins about the three orthogonal axes x , y , and z . Therefore the order-parameter manifold $\mathcal{M} = SO(3)/D_2$. This identification of the order-parameter manifold allows one to characterize the nontrivial topological excitations out of the ground state. We recall that in two spatial dimensions the fundamental group $\pi_1(\mathcal{M})$ of the order-parameter manifold \mathcal{M} is directly related to the combination law for the physical point defects [8]. More precisely, the defects are classified by the conjugacy classes of the fundamental group. To calculate the fundamental group of $\mathcal{M} = SO(3)/D_2$, one

notes that the lift of D_2 in $SU(2)$ is the eight-element non-Abelian quaternion group Q . Thus \mathcal{M} is homeomorphic to $SU(2)/Q$. Since $SU(2)$ is simply connected while Q is discrete, by using the fundamental theorem on the fundamental group, $\pi_1(\mathcal{M}) = Q$ [8]. As we will shortly see, the two-dimensional representation of Q in terms of Pauli matrices would be most relevant for our purposes. In this representation the five conjugacy classes of Q are given by

$$\begin{aligned} C_0 &= \{\emptyset\}, & \bar{C}_0 &= \{-\emptyset\}, & C_x &= \pm i\sigma_x, \\ C_y &= \pm i\sigma_y, & C_z &= \pm i\sigma_z. \end{aligned} \quad (2)$$

The class C_0 corresponds to the trivial class, i.e., no defect. The class \bar{C}_0 corresponds to a 360° disclination in two of the nematic directors, while the class C_a ($a = x, y, z$) corresponds to defects where there is a 180° disclination in all but a -axis directors.

We are interested in constructing a $T = 0$ spin-liquid state obtained by quantum disordering the nematic state. Since proliferating topological defects in quantum magnets often leads to breaking of various symmetries in the paramagnet state, one of the simplest ways to obtain a spin liquid is to destroy the nematic state *without* proliferating any topological defects. To implement this, we use an effective lattice model formulated as a gauge theory analogous to the formulation of classical spin nematics as a lattice field theory of a Z_2 gauge field coupled to a vector field [9].

The order-parameter space $\mathcal{M} = SO(3)/D_2$ is equivalent to a set of orthogonal axes $\mathbf{n}_1, \mathbf{n}_2$, and \mathbf{n}_3 at each vertex of a bigger triangular lattice with the identification $\mathbf{n}_a \equiv R\mathbf{n}_a$, where R is an element of the D_2 group. The vertices of this new triangular lattice could be taken as the centroids of the triangular plaquettes of the original triangular lattice (Fig. 1). Equivalently, by using the identification $SO(3)/D_2 \equiv SU(2)/Q$, it could be described as a quaternion gauge-matter theory whose imaginary time action is

$$\begin{aligned} S &= -t \sum z_i^* q_{ij} z_j + \kappa \left[\sum_{\Delta} \text{Tr}(q_{ij} q_{jk} q_{kl} q_{li}) \right. \\ &\quad \left. + \sum_{\square} \text{Tr}(q_{ij} q_{jk} q_{kl} q_{li}) \right] + S_B. \end{aligned} \quad (3)$$

Here $z = [z_\uparrow, z_\downarrow]^T$ is a two-component spinor with the constraint $z^\dagger z = 1$ which is minimally coupled to a quaternion gauge field q_{ij} . z 's and q 's live at the vertices and the links, respectively, of a stacked triangular lattice. The first term in the action S is the kinetic energy term for the spinons z , while the second term is the kinetic energy term for the gauge fields at a spatial (Δ) or space-time (\square) plaquette. This term penalizes topological defects and would automatically be generated from the kinetic energy term. The term S_B is the Berry phase term associated with topological defects. Since the topological defects are gapped in all the phases considered in this Letter, S_B can effectively be ignored for our purposes. The original $SO(3)$

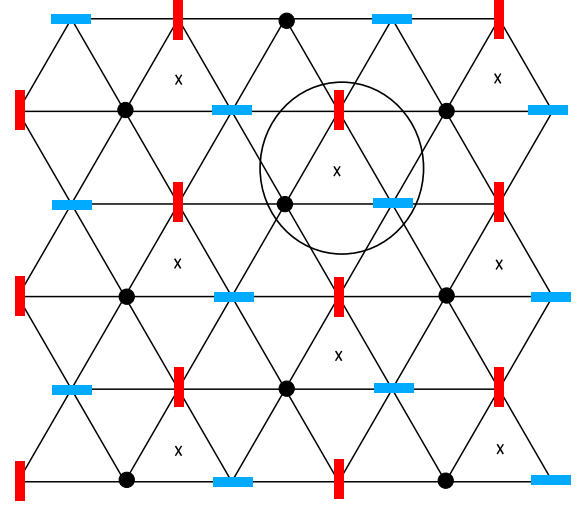


FIG. 1 (color online). The ordered state of the three-sublattice nematic state. The directors on the three sublattices are shown as three orthogonal differently colored rods (black, red, and blue). As discussed in the text, one may also define the order-parameter space in terms of spinors z , which reside at the vertices of a bigger auxiliary triangular lattice. We take the vertices of this auxiliary triangular lattice (marked as \times) to lie at the centroid of every third triangular plaquette of the original lattice.

symmetry of the spin Hamiltonian H is realized as a $SU(2)$ symmetry acting on the spinor z . This also indicates how to relate the spinor z 's to the axes $\mathbf{n}_1, \mathbf{n}_2$, and \mathbf{n}_3 . Let us define an $SU(2)$ matrix U built from z 's:

$$U = \begin{pmatrix} z_\uparrow & z_\downarrow^* \\ z_\downarrow & -z_\uparrow^* \end{pmatrix}. \quad (4)$$

The 3×3 matrix R built from U

$$R^{ab} = \frac{1}{2} \text{tr}(U^\dagger \sigma^a U \sigma^b) \quad (5)$$

has $\mathbf{n}_1, \mathbf{n}_2$, and \mathbf{n}_3 as its first, second, and third columns, respectively. Thus, all three directors \mathbf{n}_a 's are quadratic in z 's, and hence the nematic order parameter $Q_{\mu\nu}$ is of fourth degree in z 's. We note in passing that, in the context of collinear nematics, the appropriate low-energy theory is described by spin-1 “slave triplons” (i.e., three species of bosons) coupled to a Z_2 gauge field [10].

Let us describe the phase diagram corresponding to the action S in Eq. (3). First, consider S at finite temperature. Since a continuous symmetry cannot be broken in two dimensions with short-range interactions, no symmetry broken phases can exist at any nonzero temperature. Furthermore, since a pure quaternion gauge theory in two spatial dimensions is confining at any nonzero temperature, in terms of the original spin model only a paramagnet phase without any topological order can exist at $T > 0$.

Next, consider the phase diagram at zero temperature. Clearly, for $t \gg \kappa$, the spinons would condense, yielding a

three-sublattice nematic state. As discussed above, this phase would have three Goldstone modes. The non-Abelian fluxes through plaquettes would exist as excitations in this phase and correspond to the non-Abelian disclinations in Eq. (2). These fluxes have a logarithmic interaction with each other which is mediated by the spin-wave magnons. Now consider quantum disordering this phase without proliferating these defects. The resulting phase would be a paramagnet that would correspond to the deconfined phase of a pure quaternion gauge theory. The excitations in this phase would correspond to gapped spinons, quaternion fluxes, and their composites (“dyons”), which would have interesting fusion and braiding properties. Additionally, the ground state of this phase would have topological degeneracy on a torus.

The topological properties of this phase are well understood, and a detailed discussion of the particles and their topological interactions was provided by Propitius and Bais in Ref. [11]. The distinct magnetic fluxes are in one-to-one correspondence with the topological defects of the nematic phase; hence, they are classified by the conjugacy classes of the quaternion group listed in Eq. (2). As in any gauge theory, the electric charges are labeled by the irreducible representations of the gauge group. In particular, the spinons z_\uparrow and z_\downarrow transform under the two-dimensional representation of the quaternion group, while the flux composites of the form $|i\sigma_a, -i\sigma_a\rangle$ with zero net flux also carry charge and transform under the one-dimensional representation of the quaternion group. Finally, one could have dyonic particles which correspond to the bound state of a flux and an electric charge. The dyons corresponding to a flux Φ are classified by the irreducible representations of the centralizer corresponding to the conjugacy class of Φ (one may recall that the centralizer of a conjugacy class is the set of elements which commute with all elements of that conjugacy class). Since the centralizer of the \bar{C}_0 conjugacy class is the whole group Q , the bound state of a spinon and a \bar{C}_0 flux is classified by the irreducible representations of Q . On the other hand, the centralizer of the $C_x, C_y,$ and C_z conjugacy classes is the Z_4 group; hence, the corresponding dyons fall into four distinct classes corresponding to the four one-dimensional representations of the Z_4 group.

Next, consider the ground state degeneracy on a torus. Since the deconfined phase is gapped, one could calculate the degeneracy on a torus by restricting the system to a single plaquette with periodic boundary conditions. Thus, the number of ground states is proportional to the number of inequivalent flux configurations that satisfy $\prod_{\square} \sigma_{ij} \sigma_{jk} \sigma_{kl} \sigma_{li} = 1$. A simple counting shows that this number is 22. As is well known, the ground state degeneracy for a topological phase is equal to the number of distinct particle excitations of the system. Thus there must be 22 particles in a quaternion gauge-matter theory as could be easily verified by direct calculation [11].

Since spinons transform under a two-dimensional representation of the quaternion group, it is also interesting to consider the effect of transportation of a spinon around a magnetic flux. For example, when a spinon z carrying spin up,

$$z \equiv [1 \ 0]^T,$$

goes around a flux C_x , it is transformed to

$$z' = \sigma_x z = [0 \ 1]^T,$$

i.e., a spinon carrying down spin. This may seem counter-intuitive as it seemingly violates global spin conservation. Since the ground state is a spin singlet, one needs to be careful while discussing individual spinons. An instructive thought experiment which resolves this paradox is the following. Consider putting our system on a torus with a σ_x vortex threading through one of the holes of the torus. Next, an up-spinon, down-spinon pair is created out of the vacuum and the up spinon is transported around the vortex keeping the location of the down spinon fixed. Finally, the up spinon is brought back to its original location. The interesting question is whether the spinon pair remain in the spin-singlet sector (and hence can be annihilated) at the end of this process.

To answer this, we note that the gauge invariant wave function for the pair is given by $|\psi\rangle = z_{r_1}^\dagger q_{rr_1} q_{r_1 r_2} \dots q_{r_j r'} z_{r'_j}^\dagger |0\rangle$, where \mathbf{r} and \mathbf{r}' are the locations of the up and down spinon, respectively, and $q_{r_n r_{n+1}}$'s are the values of the gauge fields on the links connecting the two spinons. For concreteness, let us assume that a C_x vortex threads along the x axis and is parameterized by the gauge choice $q_{rr'} = i\sigma_x$ for $\mathbf{r} = (0, r\hat{x})$, $\mathbf{r}' = \mathbf{r} + \hat{y} \forall r$, while $q_{rr'} = 1$ otherwise. As the up spinon passes through the x axis, the field operator z_\uparrow transforms to z_\downarrow while the string operator $\mathcal{S} = q_{rr_1} q_{r_1 r_2} \dots q_{r_j r'}$ connecting the two spinons transforms as $\mathcal{S} \rightarrow \sigma_x \mathcal{S} = \sigma_x$, since in the ground state the only contribution to \mathcal{S} comes from the vortex gauge field. This implies that, at the end of the adiabatic transformation, the up spinon becomes a down spinon and vice versa since the operator \mathcal{S} acts on the down spinon and transforms it to an up spinon. Hence the spinon pair behave like an EPR spin singlet [12], and the spin is delocalized while they are separated.

Having described various possible phases of the action S , it is worthwhile to compare our approach to standard methods for obtaining spin-liquid states. Our current understanding of gapped spin liquids that do not have edge states and preserve time-reversal invariance along with lattice symmetries (“symmetric spin liquids”) draws heavily on the projective construction [1,13]. In this approach, the bosonic spin liquids are described by representing the spin- S operator in terms of a two-component boson $b = [b_1 b_2]^T$ as $S = \frac{1}{2} b^\dagger \sigma b$. The Hilbert space of the boson b is projected to the physical subspace of a spin

by the relation $b^\dagger b = 2S$. All physical operators are invariant under the $U(1)$ local transformation $b(\mathbf{r}) \rightarrow e^{i\phi(\mathbf{r})} b(\mathbf{r})$. The physical symmetries of the underlying spin Hamiltonian are realized as transformation of spinons b under these symmetries combined with a local gauge transformation. The only symmetric gapped spin-liquid states that are stable beyond the slave-particle mean-field theory consist of spinons coupled to a discrete Abelian gauge groups. One may consider other formulations such as representing S in terms of slave “triplons” (i.e., three species of bosons or fermions), though the conclusion remains unchanged [1,4,13–15]. Therefore, the non-Abelian quaternion spin liquid proposed in our Letter cannot be described by the standard slave-particle construction.

There is a more direct way to obtain the result that the standard projective construction fails to describe the quaternion spin liquid. We find that the spin operator cannot be written as an operator quadratic in spinons z for the following reason. The nematic order parameter $Q_{\mu\nu}$ is quartic in z 's, and, since it transforms as a spin-two operator, one needs to take its tensor product with itself to construct a gauge invariant physical spin operator S . Explicitly, S is given by $S_a = -i\epsilon^{abc} Q_{bd} Q_{dc}$, where the repeated indices are summed over. Therefore S would be of degree eight in z 's, which is very different than the usual slave boson theories.

The phase transition between the three-sublattice nematic phase and the non-Abelian spin liquid has also many interesting features. This phase transition corresponds to Higgs transition for the spinons z and, as we argue, is not describable within the Landau-Ginzburg-Wilson paradigm. Let us start by analyzing the symmetries of the critical action. The symmetry under physical spin rotation corresponds to the left multiplication of the matrix U [defined in Eq. (4)] by an $SU(2)$ matrix. Remarkably, the action is invariant even under the *right* multiplication of U by an $SU(2)$ matrix, and thus the critical theory has in fact $O(4) \sim SU(2) \times SU(2)$ symmetry. This is because, under the unit translation $T_{\hat{a}}$ on the triangular lattice, the \mathbf{n}_a transform as

$$T_{\hat{a}}: \mathbf{n}_1 \rightarrow \mathbf{n}_2, \quad \mathbf{n}_2 \rightarrow \mathbf{n}_3, \quad \mathbf{n}_3 \rightarrow \mathbf{n}_1. \quad (6)$$

One finds that the operation of right multiplication of U by an $SU(2)$ matrix corresponds to the rotation of \mathbf{n}_a 's among each other. The generators of these rotations K_a ($a = 1, 2, 3$) satisfy

$$[\mathbf{n}_a, K_b] = i\epsilon_{abc} \mathbf{n}_c. \quad (7)$$

Since the critical action must have both spin rotation and translational invariance, it would be invariant under the transformation $U \rightarrow V_R U V_L$, where $V_R, V_L \in SU(2)$. Hence the critical theory is

$$S = \frac{1}{g} \int d^2x \text{Tr}[(\partial_\mu U^\dagger)(\partial_\mu U)] = \frac{1}{g} \int d^2x d\tau |\partial_\mu z_\alpha|^2 \quad (8)$$

with $z^\dagger z = 1$. We emphasize that the critical theory can be written in terms of spinons z only because the topological defects are suppressed, which renders them single-valued. We note that the critical theory is very similar to that for the phase transition between a spiral antiferromagnet and a Z_2 spin liquid in a spin $S = 1/2$ triangular lattice magnet [16].

The fact that the nematic order parameter $Q_{\mu\nu}$ is of fourth degree in z has dramatic consequences for the critical correlations. For example, the critical exponent $\bar{\eta}$ defined by

$$\langle Q_{\mu\nu}(k, \omega) Q_{\mu\nu}(k, \omega) \rangle \sim \frac{1}{(\omega^2 - k^2)^{1-\bar{\eta}/2}} \quad (9)$$

would have a large value which equals $\eta = 3$ in a large N limit if one generalizes the $O(4)$ model to an $O(N)$ model. For finite N , one would obtain $\eta > 3$, whose precise numerical value we do not calculate here. This is very large compared to the anomalous exponents corresponding to the order parameter in the usual Landau-Ginzburg-Wilson theories.

Apart from the order parameter, the six conserved currents associated with the $O(4)$ symmetry would also have power-law correlations at the critical point. Three of these, the conserved total spin $S_{\text{tot}} = \sum_r S(\mathbf{r})$, are conserved microscopically, while the other three are the K_a 's defined above, which are conserved only in the low-energy effective theory.

The conserved currents acquire no anomalous dimensions and hence have scaling dimension $d = 2$. Therefore their correlations at the critical point are given by

$$\langle J_a(\mathbf{r}, \tau) J_a(0, 0) \rangle \sim \frac{1}{(r^2 + \tau^2)^2}, \quad (10)$$

where $\mathbf{J} \equiv \{S, \mathbf{K}\}$. Comparing Eq. (9) with Eq. (10), one notices that the conserved currents have a *slower* decay than that for the order parameter, which is rather unusual.

One may ask, what interactions might one add in the Hamiltonian H so as to destroy the nematic state? In the absence of any concrete Hamiltonian, we speculate that the following setting may realize stacked copies of the non-Abelian spin liquid described in this Letter. Consider a stacked triangular lattice spin $S = 1$ system and add an antiferromagnet interlayer interaction J_\perp . When $J_\perp \gg J_\parallel, K_\parallel$, the intralayer couplings, one would obtain a translationally invariant paramagnet state corresponding to decoupled spin $S = 1$ chains. On the other hand, when $J_\perp \ll J_\parallel, K_\parallel$, one obtains an ordered three-sublattice nematic state. Thus one can quantum disorder the nematic state by tuning J_\perp/J_\parallel and J_\perp/K_\parallel , and it is possible that a stacked version of our non-Abelian spin liquid may be realized in the intermediate regime.

In summary, we have described a non-Abelian spin liquid in a spin $S = 1$ quantum magnet on a triangular lattice which cannot be accessed within the standard slave boson or fermion projective construction. The non-Abelian phase has interesting topological features captured by the ground state degeneracy on a torus and braiding and fusion of its excitations. We also described a non-Landau phase transition between this spin liquid and a noncollinear nematic state. The nematic correlations near this phase transition are characterized by a large anomalous dimension.

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