

Symplectic properties of Milnor fibres

by

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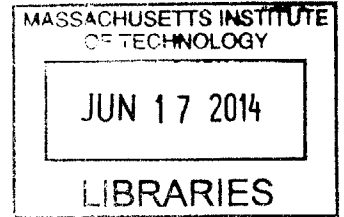
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Abstract

We present two results relating to the symplectic geometry of the Milnor fibres of isolated affine hypersurface singularities.

First, given two Lagrangian spheres in an exact symplectic manifold, we find conditions under which the Dehn twists about them generate a free non-abelian subgroup of the symplectic mapping class group. This extends a result of Ishida for Riemann surfaces. The proof generalises the categorical version of Seidel's long exact sequence to arbitrary powers of a fixed Dehn twist. We also show that the Milnor fibre of any isolated degenerate hypersurface singularity contains such pairs of spheres.

In the second half of this thesis, we study exact Lagrangian tori in Milnor fibres. The Milnor fibre of any isolated hypersurface singularity contains many exact Lagrangian spheres: the vanishing cycles associated to a Morsification of the singularity. Moreover, for simple singularities, it is known that the only possible exact Lagrangians are spheres. We construct exact Lagrangian tori in the Milnor fibres of all non-simple singularities of real dimension four. This gives examples of Milnor fibres whose Fukaya categories are not generated by vanishing cycles. Also, this allows progress towards mirror symmetry for unimodal singularities, which are one level of complexity up from the simple ones.

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Contents

1	Introduction	9
1.1	Dehn twists and free subgroups of the symplectic mapping class group: overview	10
1.1.1	Outline	12
1.2	Lagrangian tori in four-dimensional Milnor fibres: overview	14
1.2.1	Constructing the Lagrangian torus T in the Milnor fibre of $T_{p,q,r}$: a sketch	17
1.2.2	Extension to higher dimensions: Lagrangian $S^1 \times S^{n-1}$'s	18
1.2.3	Contents	19
1.2.4	Conventions	19
2	Dehn twists and free subgroups of the symplectic mapping class group	21
2.1	A_∞ categories and twisted complexes thereof: preliminaries	21
2.1.1	Definitions and notation	21
2.1.2	Choosing ‘nice’ representatives for A_∞ categories	22
2.1.3	Background on twisted complexes	22
2.2	The Fukaya category of an exact manifold: preliminaries	26
2.2.1	Floer cohomology	27
2.2.2	The Fukaya category	28
2.2.3	A_∞ algebra associated to a Lagrangian sphere.	29
2.2.4	Isomorphism in the Fukaya category: some criteria	31
2.3	Classification of finite dimensional A -modules	34
2.4	Twisted complexes and iterations of Dehn twists	37
2.5	Lower bounds on $hf(\tau_L^n L_0, L_1)$	42
2.5.1	$T_L^n L_0$ as a cone	42
2.5.2	Quasi-isomorphic truncated bar complexes	42
2.5.3	Rank Inequalities	43
2.6	Conclusion of argument	46
2.7	Examples: some Milnor fibres	49
2.7.1	Milnor fibres as symplectic manifolds	49
2.7.2	Adjacency of singularities and symplectic embeddings	52
2.7.3	Lagrangian spheres in (A_m) fibres	53
2.7.4	Some families of examples	54
3	Lagrangian tori in four-dimensional Milnor fibres	59
3.1	Background on singularity theory	59
3.1.1	Basic properties	59

3.1.2	Some Picard-Lefschetz theory	65
3.1.3	Milnor fibres of functions of two variables, following A'Campo	71
3.1.4	Distinguished bases for $P(x_0, \dots, x_n) + x_{n+1}^{d+1}$	74
3.1.5	Some extensions	78
3.2	Background on Floer theory and Fukaya categories	79
3.2.1	What flavour of the Fukaya category do we use?	79
3.2.2	Generation and split-generation	81
3.2.3	Convexity arguments	82
3.3	Vanishing cycles for $T_{p,q,r}$	82
3.3.1	Case of $T_{p,q,2}$	82
3.3.2	Case of $T_{3,3,3}$	86
3.3.3	Case of a general $T_{p,q,r}$	93
3.4	Torus construction	97
3.4.1	Lagrangian surgery	97
3.4.2	Main construction	98
3.4.3	A local model for the Lagrangian surgeries	101
3.4.4	Tori in parabolic singularities	102
3.5	The Fukaya category of unimodal Milnor fibres	108
3.5.1	Statements of results	108
3.5.2	Floer cohomology of T with the vanishing cycles P_1 , Q_1 and R_1	109
3.5.3	Floer cohomology of T with the vanishing cycles A and B	112
3.6	Mirror symmetry for $T_{p,q,r}$	114
3.6.1	The derived category of coherent sheaves on an orbifold \mathbb{P}^1	114
3.6.2	The derived directed Fukaya category of $T_{p,q,r}$	114
A Visualizing the singular values of $M(x, y, z; t)$		119

Chapter 1

Introduction

Consider a hypersurface in \mathbb{C}^n with isolated critical points. Consider a neighbourhood of one of them. How can we understand the geometry of the hypersurface in that neighbourhood? One approach is to use the smoothing given by a small deformation of the defining equation of the hypersurface. The result is called the Milnor fibre of the singularity, and the landmark text [51] provides an elegant answer to the above question. More broadly, Milnor fibres are central protagonists in classical singularity theory, as presented for instance in the excellent survey [7].

As open subsets of affine hypersurfaces, Milnor fibres carry a symplectic form given by restricting the usual Kaehler form on \mathbb{C}^n . This symplectic form, say ω , is exact: $\omega = d\theta$, for some one-form θ . Milnor fibres are interesting symplectic objects of their own right. Moreover, using embeddings, many results about their geometry extend to a larger class of symplectic manifolds, notably affine hypersurfaces. From a different perspective, questions about their geometry also tie into the following suggestion of Arnol'd, in the Floer Memorial Volume [11]: that symplectic geometry would provide a rich framework in which to revisit and strengthen classical singularity theory, many of whose results concern smooth or topological categories.

The present thesis provides information about the following two questions:

- *What are the symplectomorphisms of a Milnor fibre?*
- *What Lagrangian submanifolds can a Milnor fibre have?*

Symplectic Picard–Lefschetz theory provides the Milnor fibre of a singularity f with a distinguished family of Lagrangian spheres: vanishing cycles associated to f . Moreover, there are distinguished symplectomorphisms: the Dehn twists about those vanishing cycles. In particular, the monodromy of f is the product of Dehn twists about a distinguished collection of spheres.

Work towards the first question is given in Chapter 2. We consider the group of compactly supported symplectomorphisms of a Milnor fibre M up to isotopy: $\pi_0(\text{Sym}^c M)$. In the case where the Milnor fibre has real dimension two, this is the mapping class group of a Riemann surface, which has many known structural properties. In particular, it is generated by Dehn twists about vanishing cycles. Ishida [35] gives conditions under which there are no relations in the mapping class group between two Dehn twists; we generalise this to higher dimensions. Our main theorem (1.1.1) applies to a broader set of symplectic manifolds, and Milnor fibres

provide interesting collections of examples. For a further introduction to this chapter, and an outline of the arguments, see 1.1. A version of this work has appeared in [40].

Work towards the second question is given in Chapter 3. We focus on the case of Milnor fibres of real dimension four, and on *exact* Lagrangians L : $[\theta|_L] = 0 \in H^1(L)$. The reader might think of this as an additional rigidity condition. These give objects of the Fukaya category of the Milnor fibre, and morphisms are Floer cohomology groups. (In the case at hand, the analytic foundations of these tools are already well-defined.) In the case of simple singularities, which are associated to Dynkin diagrams of type ADE , it is known that the only possible exact Lagrangians are spheres (this follows from [56] combined with topological arguments). We show that the Milnor fibres of any of the other singularities contain an exact Lagrangian torus. Moreover, for an infinite family of examples called the $T_{p,q,r}$ singularities, we use the torus to show that the Fukaya category of the Milnor fibre is not generated by vanishing cycles: our tori are novel algebraically as well as geometrically. Along the way, we also get an answer to a mirror symmetry question for $T_{p,q,r}$. For a further introduction to this chapter, see Section 1.2.

1.1 Dehn twists and free subgroups of the symplectic mapping class group: overview

Given any Lagrangian sphere L in a symplectic manifold M , one can define the Dehn twist τ_L about it. This is a symplectomorphism supported in a tubular neighbourhood of L , that is defined up to Hamiltonian isotopy, and a choice of framing for the sphere ([59]). If M and L are exact, so is the symplectomorphism. These are sometimes known as generalised Dehn twists; in real dimension two, they recover the classical notion of a Dehn twist on a Riemann surface (discussed e.g. in [24]). They arise, for instance, when studying the monodromy of a Milnor fibration, thought of as a compactly supported symplectomorphism. For a Morse singularity, this automorphism is a Dehn twist. For a general isolated hypersurface singularity, it can be expressed as a composition of Dehn twists about a collection of Lagrangian spheres. A natural problem is to study the subgroup of symplectomorphisms (up to some isotopy) generated by these. More generally, given a symplectic manifold with a set of distinguished Lagrangian spheres, one might want to understand the automorphisms generated by the Dehn twists about them; we study a sub-question.

Fix two Lagrangian spheres, say L and L' . If they are disjoint, the two Dehn twists commute. If they have exactly one intersection point, the Dehn twists have a braid relation: $\tau_L \tau_{L'} \tau_L = \tau_{L'} \tau_L \tau_{L'}$. (One can check this manually for Riemann surfaces. In general, it can be derived from monodromy computations for (A_2) type singularities; see [63] and Appendix A of [59].) Generically, it seems there need not be any other relations [14, 42]. We consider the case where the two spheres intersect twice or more; Ishida has studied this for Riemann surfaces.

Theorem (Ishida [35]). *Suppose a, b are a pair of simple closed curves on $\Sigma_{g,n}$, an n -punctured genus g surface, and that $I_{\min}(a, b) \geq 2$. Then there are no relations between τ_a and τ_b .*

$I_{\min}(a, b)$ is the minimal intersection number of a and b , as defined in [24]; it is the smallest unsigned count of intersection points, varying over the isotopy classes of a and b . The key

technical lemma for Ishida's result is the inequality

$$|n| \cdot I_{\min}(c, a) \cdot I_{\min}(a, b) \leq I_{\min}(c, b) + I_{\min}(\tau_a^n(c), b) \quad (1.1.1)$$

that holds for any triple of simple closed curves a , b and c . To generalise this result to higher dimensions, we need to find a suitable measure of 'intersection'. Suppose that M is an exact symplectic manifold with contact type boundary, L a Lagrangian sphere, and L_0, L_1 arbitrary compact exact Lagrangians. Seidel's long exact sequence [61] gives the inequality of Floer ranks

$$\dim(HF(L_1, L)) \cdot \dim(HF(L, L_0)) \leq \dim(HF(L_1, L_0)) + \dim(HF(\tau_L(L_1), L_0)). \quad (1.1.2)$$

This suggests using rank of Floer cohomology groups as a substitute for intersection numbers; these recover the minimal intersection number of any two non-isotopic simple closed curves. However, for any exact Lagrangian sphere L and any Hamiltonian perturbation ϕ , $rk(HF(L, \phi(L))) = 2$, and Dehn twists about these spheres are isotopic. We have to make sure to exclude such cases. Such a pair L and $\phi(L)$ are Lagrangian isotopic, which in particular implies that they are quasi-isomorphic objects of the Fukaya category of M , which in this case just means that the Floer products

$$HF(L, \phi(L)) \otimes HF(\phi(L), L) \rightarrow HF(\phi(L), \phi(L)) \quad (1.1.3)$$

$$HF(\phi(L), L) \otimes HF(L, \phi(L)) \rightarrow HF(L, L) \quad (1.1.4)$$

are surjective (see Corollary 2.2.6). Our main result is:

Theorem 1.1.1. *Suppose $n \geq 2$. Let M^{2n} be an exact symplectic manifold with contact type boundary, and $\pi_0(\text{Sympl}(M))$ the group of symplectomorphisms of M , up to symplectic isotopy. Suppose L and L' are two Lagrangian spheres such that $\dim(HF(L, L')) \geq 2$; additionally, if $\dim(HF(L, L')) = 2$, we require that L and L' be not quasi-isomorphic in the Fukaya category of M . Then the Dehn twists τ_L and $\tau_{L'}$ generate a free subgroup of $\pi_0(\text{Sympl}(M))$.*

A Dehn twist induces a functor from the Fukaya category of M , $\mathcal{Fuk}(M)$, to itself, defined up to quasi-isomorphism; it is invertible, also up to quasi-isomorphism. In particular, the induced functor on the cohomology category $H(\mathcal{Fuk}(M))$ is invertible. We use a stronger form of the long exact sequence of [61] that is phrased using twisted complexes, viewing the Dehn twist as such a functor [62, Corollary 17.17]. The proof of Theorem 1.1.1 hinges on a generalisation of this to arbitrary powers of a fixed Dehn twist (Proposition 2.4.4). As a by-product we show that:

Theorem 1.1.2. *τ_L and $\tau_{L'}$ generate a free subgroup of automorphisms of $H(\mathcal{Fuk}(M))$.*

Remark 1.1.3. *The hypotheses on M , L and L' under which we prove Theorem 1.1.1 are actually weaker, but more technical. Note that if L and L' are Lagrangian isotopic, they are quasi-isomorphic in the Fukaya category. As we heavily use Floer and Fukaya-theoretic tools, which cannot tell quasi-isomorphic elements apart, it seems unlikely that we will be able to relax that condition without drastically modifying our approach.*

We give examples of such pairs of Lagrangian spheres in the Milnor fibres of all degenerate isolated hypersurface singularities. For some of these, we also arrange for the spheres to be

homologous, in which case the result cannot be at all detected using the Picard-Lefschetz theorem, or other homological tools. Such examples exist for all degenerate isolated singularities in the even complex-dimensional case. In the odd-dimensional case, though we cannot find any for (A_2) , they exist at least for all singularities adjacent to (A_3) (see section 3.1.1 for a definition).

Suppose M satisfies the hypothesis of Theorem 1.1.1. Let $\pi_0(\text{Diff}^+M)$ be the group of orientation-preserving diffeomorphisms of M , up to smooth isotopy. There is a natural map

$$\pi_0(\text{Sym}p(M)) \rightarrow \pi_0(\text{Diff}^+M). \quad (1.1.5)$$

What can we say about its kernel? Whenever $\dim_{\mathbb{R}}M = 4$, one has $\tau_L^2 = \text{Id} \in \pi_0(\text{Diff}^+M)$ [59, Lemma 6.3]. Moreover, whenever $\dim_{\mathbb{R}}M = 2n$ for n even, we have that $\tau_L^{k_n} = \text{Id} \in \pi_0(\text{Diff}^+M)$, for some integer k_n [44, Section 3]. We then have the following immediate corollary to Theorem 1.1.1.

Corollary 1.1.4. *Suppose M^{2n} satisfies the hypothesis of Theorem 1.1.1, and n is even. Then the kernel of the forgetful map $\pi_0(\text{Sym}p(M)) \rightarrow \pi_0(\text{Diff}^+M)$ contains a free non-abelian subgroup, generated by $\tau_L^{k_n}$ and $\tau_{L'}^{k_n}$.*

Additionally, for all the examples of homologous L, L' that we construct in Milnor fibres, we show either that they are smoothly isotopic (even-dimensional case), or, in the odd complex-dimensional case, that they can be made disjoint after a smooth isotopy. Thus τ_L and $\tau_{L'}$ commute (or even agree) as elements of $\pi_0(\text{Diff}^+M)$, and the kernel again has a free non-abelian subgroup.

1.1.1 Outline

The main calculation involves twisted complexes associated to the Fukaya category of M , an A_∞ category. Accordingly, section 2.1 collects some background material on A_∞ categories, and twisted complexes thereof. Section 2.2 does the same for the Fukaya category of an exact symplectic manifold with contact type boundary. The first noteworthy ingredient is:

- Let L a Lagrangian sphere, and A be the A_∞ algebra on two generators given by $\mathbb{Z}_2[\epsilon]/(\epsilon^2)$, with $\mu^2(1, \epsilon) = \mu^2(\epsilon, 1) = \epsilon$, and all other A_∞ structure maps trivial. The Floer chain group $CF(L, L)$ with its Fukaya A_∞ structure is quasi-isomorphic to A . (Proposition 2.2.3.)

This enables us to obtain two technical criteria for quasi-isomorphisms of Lagrangians, both of which feed into the main proof:

- Suppose that L_0 and L_1 are Lagrangian spheres, that $\dim(HF(L_0, L_1)) = 2$, and that multiplication

$$HF(L_0, L_1) \otimes HF(L_1, L_0) \rightarrow HF(L_1, L_1)$$

is surjective. Then L_0 and L_1 are quasi-isomorphic in the Fukaya category. (Corollary 2.2.6.)

- Take two Lagrangian spheres L_0, L_1 . Let $\epsilon \in HF(L_0, L_0) \cong H^*(L_0; \mathbb{Z}/2)$ be the generator corresponding to the top degree cohomology class. Consider its action by product

on $HF(L_0, L_1)$, which is the map

$$HF(L_0, L_1) \rightarrow HF(L_0, L_1) \quad (1.1.6)$$

$$[a] \mapsto [\mu_2(a, \epsilon)]. \quad (1.1.7)$$

If it is nonzero, then L_0 and L_1 are quasi-isomorphic in the Fukaya category. (Corollary 2.2.9.)

The main body of the proof is in sections 2.4 and 2.5. First, some notation: let L be a Lagrangian sphere, and L_0, L_1 any exact Lagrangians. $hom(L, L_0)$ denotes morphisms in the Fukaya category, and ev evaluations maps, for instance $hom(L, L_0) \otimes L \rightarrow L_0$.

Let (ϵ) be the one-dimensional \mathbb{Z}_2 -vector space generated by ϵ . Iterated applications of the long exact sequence of [61] give an expression for $\tau_L^n L_0$; we start by simplifying this to show that:

Proposition 2.4.4. $\tau_L^n L_0$ *it is quasi-isomorphic to the twisted complex*

$$L_0 \oplus hom(L, L_0) \otimes L \oplus hom(L, L_0) \otimes (\epsilon) \otimes L \oplus \dots \oplus hom(L, L_0) \otimes \overbrace{(\epsilon) \dots (\epsilon)}^{n-1} \otimes L \quad (1.1.8)$$

with differential acting on the r^{th} summand by

$$Id^{\otimes r-2} \otimes ev \oplus \sum_{r=i+j+k, j>1} Id^{\otimes i} \otimes \mu_{\mathcal{A}}^j \otimes Id^{\otimes k-1} \otimes 1. \quad (1.1.9)$$

This requires a weak form of Proposition 2.2.3: we only need the structure on the cohomology level. The twisted complex (1.1.8) immediately gives an expression for $hom(L_1, \tau_L^n L_0)$. Consider the A_∞ reduced bar complex corresponding to the tensor over A of $hom(L, L_0) =: M$ and $hom(L_1, L) =: N$, say

$$\begin{aligned} M \otimes_A N := \\ (M \otimes N) \oplus (M \otimes (\epsilon) \otimes N) \oplus (M \otimes (\epsilon) \otimes (\epsilon) \otimes N) \oplus \dots \oplus (M \otimes (\epsilon) \otimes \dots \otimes (\epsilon) \otimes N) \oplus \dots \end{aligned} \quad (1.1.10)$$

where tensor products without subscripts are taken over \mathbb{Z}_2 , and the differential is given by summing over all the possible $Id^{\otimes i} \otimes \mu_{\mathcal{A}}^j \otimes Id^{\otimes k}$. (See [45, Section 2.3.3].) We shall denote by $(M \otimes_A N)_n$ the truncation of complex (1.1.10) that consists of only the first n summands. We find that

$$hom(L_1, \tau_L^n L_0) = hom(L_1, L_0) \oplus (hom(L, L_0) \otimes_A hom(L_1, L))_n \quad (1.1.11)$$

with the obvious differential, again obtained by taking all possible A_∞ products. Moreover, we get the inequality

$$dim(HF(\tau_L^n L_0, L_1)) + dim(HF(L_0, L_1)) \geq dim((hom(L, L_0) \otimes_A hom(L_1, L))_n). \quad (1.1.12)$$

This is used to prove weak analogs of the inequality used by Ishida, by estimating the right-hand side. In order to do so, we first classify finite dimensional A_∞ left and right modules

over A (Proposition 2.3.3), of which $\text{hom}(L, L_0)$ and $\text{hom}(L_1, L)$ are examples. This implies the following:

Proposition 2.5.4. *For all integers $n \neq 0$, we have*

$$\dim(\text{HF}(\tau_L^n(L_0), L_1)) + \dim(\text{HF}(L_0, L_1)) \geq \dim(\text{HF}(L, L_1)) \cdot \dim(\text{HF}(L_0, L)). \quad (1.1.13)$$

Further, if $L \not\cong L_0, L_1$ in the Fukaya category, and $|n| \geq 2$, we find that

$$\dim(\text{HF}(\tau_L^n(L_0), L_1)) + \dim(\text{HF}(L_0, L_1)) \geq 2 \dim(\text{HF}(L, L_1)) \cdot \dim(\text{HF}(L_0, L)). \quad (1.1.14)$$

Inequality (1.1.14) also uses Corollary 2.2.9. Inequalities (1.1.13) and (1.1.14) are strong enough to allow us to conclude the proofs of Theorems 1.1.1 and 1.1.2, in section 2.6, by following Ishida’s argument [35]. The case $\dim(\text{HF}(L, L')) = 2$ requires careful consideration, and additionally, the use of Corollary 2.2.6.

Section 2.7 contains the examples. We consider Milnor fibres of isolated hyperplane singularities with the natural symplectic form. Classically, if a singularity is adjacent to another, one gets a smooth embedding of one Milnor fibre into another; there is an analogous fact in the symplectic setting. By a result of Abouzaid (Lemma 2.2.4), the structure of the Floer complex between two Lagrangians is preserved under such an embedding. This implies that if a singularity $[f]$ is adjacent to $[g]$, and the Milnor fibre of g contains Lagrangian spheres satisfying the hypothesis of Theorem 1.1.1, then so do the Milnor fibres of representatives of $[f]$ (Corollary 2.7.10). It is thus enough to construct strategically positioned such spheres, in the Milnor fibre of the (A_2) singularity (and, to get the homologous odd-dimensional examples, (A_3)). We do this by hand, using the framework of Khovanov and Seidel [42]. In the even dimensional case, we shall get ‘for free’ that the two spheres are smoothly isotopic. In the odd dimensional case, we show that one of the spheres that we construct in the (A_3) fibre can be isotoped so as not to intersect the other using the Whitney trick.

1.2 Lagrangian tori in four-dimensional Milnor fibres: overview

Let $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ be a holomorphic function. Suppose its differential has an isolated zero at the origin: $df|_0 = 0$, but $df \neq 0$ on some punctured open ball $B_r^*(0)$. An *isolated hypersurface singularity* is the equivalence class of the germ of such an f , up to holomorphic re-parametrisation. Assume for simplicity that $f(0) = 0$. The *Milnor fibre* of f , studied in [51], is the smooth manifold

$$M_f := f^{-1}(\epsilon_\delta) \cap B_\delta(0) \quad (1.2.1)$$

for suitably small δ and ϵ_δ . This carries an exact symplectic structure, say $\omega = d\theta$, inherited from \mathbb{C}^{n+1} .¹

Perturb f generically, say to \tilde{f} . The singularity at zero splits into a collection of complex Morse singularities near zero. As we’ve chosen an isolated singularity, there are finitely many of them; their count is called the *Milnor number* of f . Fix a regular value of \tilde{f} , say a , and a collection of paths between each of the singular values and a . The fibre of \tilde{f} above a is itself a

¹The exact symplectic manifold in equation 1.2.1 depends, a priori, on several choices. One can always attach cylindrical ends to its boundary, which gives a (non-compact) Liouville domain. We shall see that this Liouville domain is independent of choices, including holomorphic re-parametrisation – see Lemma 3.1.5.

copy of the Milnor fibre of f . each path determines a Lagrangian sphere in it: the vanishing cycle associated to that path. Topologically, this already gives much information [51]: indeed, M_f is homotopic to a wedge of half-dimensional spheres:

$$M_f \cong \bigvee_{\mu} S^n \tag{1.2.2}$$

where μ is the Milnor number of f , and a basis for $H_n(M_f; \mathbb{Z})$ is given by a distinguished collection of vanishing cycles. One immediate consequence is that there is a large supply of Lagrangian spheres in Milnor fibres. For $n \geq 2$, any such sphere L is automatically exact: the cohomology class $[\theta|_L] = 0 \in H^1(L)$ vanishes. We are interested in the following question:

What are the possible exact Lagrangian submanifolds in a Milnor fibre?

We will focus on compact Lagrangians, and the case $n = 2$, which means the Milnor fibre has real dimension four. For a word about higher dimensions, see Section 1.2.2. Why do we require exactness? Non-exact Lagrangians are much easier to come by, notably tori in Darboux charts. It could also be interesting to consider Lagrangians with different forms of rigidity requirements – e.g. ones with self-Floer cohomology defined and non-zero; however, we shall not address such questions here.

Particular attention has been paid to the symplectic geometry of a distinguished collection of singularities, known as *simple* or *ADE-type* singularities – for a far from exhaustive sample of the flavour of questions studied, see e.g. [42, 36, 23, 69, 16]. Any isolated hypersurface singularity f has an invariant called its *intersection form*. In the case $n = 2$, it agrees with the usual intersection form on $H_2(M_f; \mathbb{Z})$; our orientations are chosen such that for any compact Lagrangian $L \subset M_f$, we have $L \cdot L = -\chi(L)$.

Classically, one criterion that distinguishes simple singularities is that they are precisely the ones whose the intersection form is negative definite [65]. Together with work of Ritter [56], this implies that the only possible exact Lagrangians are spheres. For all other singularities, the intersection form is semi-definite or indefinite. In particular, in the case $n = 2$, this leaves room for tori. Our first result is that these exist:

Theorem. *(Theorem 3.4.7.) The Milnor fibre of any non-simple isolated hypersurface singularity of three variables contains an exact Lagrangian torus T , primitive in homology, and with vanishing Maslov class.*

The vanishing of the Maslov class is useful from the perspective of Floer theory, as it allows one to use a Fukaya category with absolute \mathbb{Z} -gradings. (See Section 3.2.)

The simple singularities have *modality zero*, whereas all other singularities have positive modality. Loosely speaking, the modality of a singularity f is the dimension of a parameter space covering a neighbourhood of f in the space of singularities *after* holomorphic reparametrization (see Definition 3.1.8). This means that suitably interpreted, the non-simple singularities are generic.

Our approach is to construct tori explicitly in strategically chosen Milnor fibres, and use embeddings from these Milnor fibres to get tori in all others. These embeddings are geometric consequences of a phenomenon known as *adjacency* of singularities. See Section 3.1.1.

The bulk of this article focuses on the singularities for which we construct T explicitly. In the classification of Arnol'd [7], they are known as $T_{p,q,r}$ singularities. They are of the form

$$T_{p,q,r}(x, y, z) = x^p + y^q + z^r + axyz \quad (1.2.3)$$

where p, q and r are integers such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1 \quad (1.2.4)$$

and $a \in \mathbb{C}$ is a complex parameter, which is allowed to take all but finitely many values for each triple (p, q, r) . In the case where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, the condition is $a \neq 0$. For these singularities, we have the following further properties:

- There exists an exact Lagrangian torus with vanishing Maslov class in every primitive class in the nullspace of the intersection form (Theorem 3.4.11).
- We compute Floer cohomology between our torus and every vanishing cycle in a distinguished collection (Proposition 3.5.1). In particular, if we equip our torus with any spin structure and a generic complex flat line bundle, all of those Floer groups vanishes.

Remark 1.2.1. *If one only wanted to prove Theorem 3.4.7, it would have been enough to consider the cases $(p, q, r) = (3, 3, 3), (2, 4, 4)$ and $(2, 3, 6)$. Note these are the three triples of integers for which $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.*

As a consequence of Proposition 3.5.1, we show the following:

Theorem. *(Theorem 3.5.3) The Fukaya category of the Milnor fibre of $T_{p,q,r}$ is not split-generated by any collection of vanishing cycles.*

In contrast, for all the previously understood examples of Milnor fibres, the Fukaya category is generated by vanishing cycles. These examples include simple singularities, and most weighted homogeneous singularities – see Proposition 3.2.3, due to Seidel.

Our key technical result is a detailed geometric description of the Milnor fibre of $T_{p,q,r}$, together with a distinguished collection of vanishing cycles (Proposition 3.3.3). In particular, it also gives enough information to answer a mirror symmetric question about $T_{p,q,r}$:

Theorem. *(Theorem 3.6.1) There is an equivalence*

$$D^b \mathcal{Fuk}^{\rightarrow}(T_{p,q,r}) \cong D^b \mathcal{Coh}(\mathbb{P}^1_{p,q,r}) \quad (1.2.5)$$

where the left-hand side is the bounded derived directed Fukaya category of the singularity $T_{p,q,r}$, and the right-hand side is the bounded derived category of coherent sheaves on an orbifold \mathbb{P}^1 , with orbifold points with isotropies of order p, q and r .

For more information, see Section 3.6. The directed Fukaya category of $T_{p,q,r}$ contains less information than the ‘full’ Fukaya category, which was the one that we considered in Theorem 3.5.3. We hope our techniques will enable us to understand a version of mirror symmetry for this too. Already, Theorem 3.6.1 complements existing results in the literature. In particular, it provides an answer to Conjecture 1 of [22] and Conjecture 7.4 of [64]; both of

these articles consider mirror-symmetric questions for these singularities, studying algebraic invariants. Among other works, Ueda [67] proves a related statement when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, using the fact that $T_{p,q,r}$ is weighted homogeneous in those cases. Recent work of Cho, Hong, Kim and Lau [18, 19] studies the same spaces, but with the A and B -sides swapped: they take $\mathbb{P}^1_{a,b,c}$ as the A -side. They show that its derived Fukaya category corresponds to matrix factorizations of a Landau–Ginzberg potential whose leading terms agree with $T_{p,q,r}$.

1.2.1 Constructing the Lagrangian torus T in the Milnor fibre of $T_{p,q,r}$: a sketch

Proposition 3.3.3 describes the Milnor fibre of $T_{p,q,r}$ as an open subset of the total space of a Lefschetz fibration; vanishing cycles for $T_{p,q,r}$ are given by matching paths for that fibration. In the example of $T_{3,4,5}$, here is what we get. The basis of the fibration is given in Figure 1-1, with matching paths given in colour. The smooth fibre is a three-punctured surface of genus four. In the fibre above \star , where most of the matching paths meet, the matching cycles restrict to the curves of Figure 1-2.

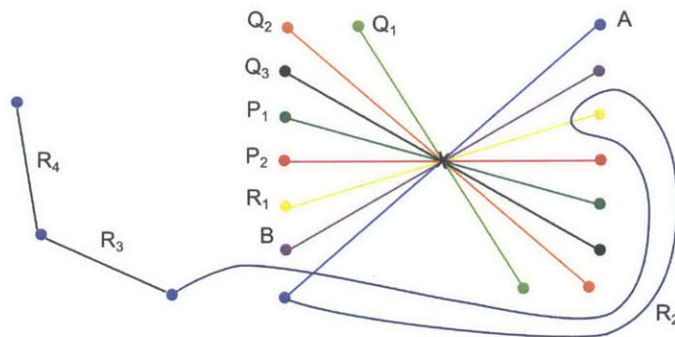


Figure 1-1: Basis of the Lefschetz fibration used to describe $T_{3,4,5}$.

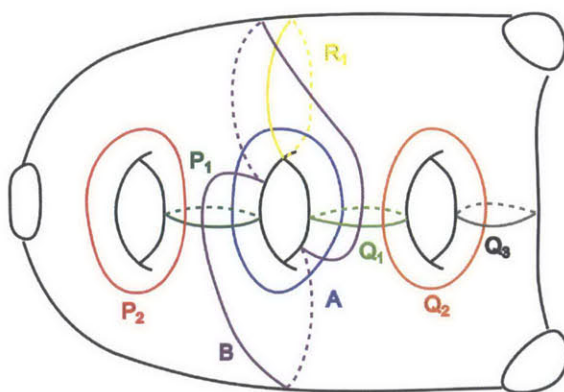


Figure 1-2: Fibre above \star in the Lefschetz fibration used to describe $T_{3,4,5}$.

The P_i 's and Q_i 's form chains of lengths $p - 1$ and $q - 1$ in the fibre direction. The R_i 's form a chain in the base direction. The spheres A and R_2 intersect twice, with opposite

orientations. After Hamiltonian isotopy, we can arrange for them not to intersect (Lemma 3.3.2); all other intersections are already minimal. This description also recovers Gabriellov's description of the intersection form of $T_{p,q,r}$ [30, 31]. We use three tools:

- A symplectic version of work of A'Campo [5], which gives an algorithm for describing the Milnor fibre of a function of two variables, together with a distinguished collection of vanishing cycles. See Section 3.1.3.
- A Thom-Sebastiani-type technique, based on work of Gabriellov [30], which, given a distinguished collection of vanishing paths and cycles for $f(x_0, \dots, x_n)$, gives one for $f(x_0, \dots, x_n) + x_{n+1}^d$. See Section 3.1.4.
- Deformation arguments, which allow us to embed the Milnor fibres we care about into the fibres of functions of the form $g(x, y) + z^d$, which are covered by the techniques above. We use deformation arguments when $r \geq 3$. See Sections 3.3.2 and 3.3.

Remark 1.2.2. *We use a deformation argument for $T_{3,3,3}$, even though one could get a description of its Milnor fibre and vanishing cycles using solely the first two techniques: one representative for $T_{3,3,3}$ is $x^3 + y^3 + z^3$. The reasons for using a deformation are two-fold: one the one hand, it readily gives a configuration in which we can perform surgery to get an exact torus, and on the other hand, it allows us to extend to the case of higher r .*

The spheres A and B of Figure 1-1 and 1-2 intersect in two points, with agreeing orientation. Our torus T is obtained by performing Lagrangian surgery [55] at each of those two points. Note that in general, such a construction would not preserve exactness. The fact that we have that, as well as the vanishing of the Maslov class of T , are consequences of different geometric features of the Milnor fibre. Details are in Section 3.4.3.

To calculate Floer cohomology groups between T and the vanishing cycles, or the A_∞ -products in the directed Fukaya category $\mathcal{Fuk}^\rightarrow(T_{p,q,r})$, we reduce the problem to counting holomorphic curves on the Riemann surface that is the preimage of \star . In the case involving T , this requires a little care; see Section 3.5.

1.2.2 Extension to higher dimensions: Lagrangian $S^1 \times S^{n-1}$'s

The present article focuses on the case when M_f has real dimension four. However, let us make the following remark about the higher-dimensional case: Starting with our description of the Milnor fibre of

$$x^p + y^q + z^r + axyz$$

one gets a description of the Milnor fibre of its stabilization to a function of more variables, also known as $T_{p,q,r}$:

$$x_0^p + x_1^q + x_2^r + ax_0x_1x_2 + x_3^2 + \dots + x_{n-2}^2.$$

In particular, there will be two n -dimensional vanishing cycles, A' and B' , which intersect in two points with agreeing orientation. Performing surgery at those two points, one gets a Lagrangian $S^1 \times S^{n-1}$. One can check that the argument for exactness in the $n = 2$ case readily extends here. Moreover, the result we use about adjacency of singularities also holds for singularities of more variables. Thus we have the following:

Proposition 1.2.3. *In higher dimensions, the Milnor fibre of any singularity of positive modality contains an exact Lagrangian $S^1 \times S^{n-1}$, primitive in homology.*

1.2.3 Contents

Section 3.1 collects material on singularity theory, principally from a symplectic perspective. In particular, we re-visit the methods of A'Campo (Subsection 3.1.3) and Gabrielov (Subsection 3.1.4) for studying Milnor fibres and vanishing cycles, and set them up in a symplectic framework. Section 3.2 gives some background on the version of the Fukaya category that we use, and relevant properties. The description of the Milnor fibre of $T_{p,q,r}$ together with a distinguished collection of vanishing cycles is in Section 3.3. Section 3.4.3 constructs tori in these Milnor fibres, and proves Theorem 3.4.7 and Proposition 3.4.11; it also presents a useful local model for the construction. Section 3.5 gives the results relating to Floer cohomology between the torus we construct and the vanishing cycles of the $T_{p,q,r}$, including Theorem 3.5.3 relating to generation of the Fukaya categories of the Milnor fibres of these singularities. Finally, Section 3.6 proves Theorem 3.6.1 on homological mirror symmetry for $T_{p,q,r}$.

1.2.4 Conventions

All singular (co)homology groups have coefficients in \mathbb{Z} unless otherwise specified.

Chapter 2

Dehn twists and free subgroups of the symplectic mapping class group

2.1 A_∞ categories and twisted complexes thereof: preliminaries

2.1.1 Definitions and notation

Our A_∞ categories are defined over the field of two elements \mathbb{Z}_2 , and do not have gradings. An A_∞ category \mathcal{A} will consist of a set of objects $\text{Ob}\mathcal{A}$, and, for each ordered pair of objects (L_0, L_1) , a \mathbb{Z}_2 -vector space of morphisms between them, denoted by $\text{hom}_{\mathcal{A}}(L_0, L_1)$. This is equipped with additional structure, the A_∞ composition maps. We follow [62]’s convention on the ordering of indices for these:

$$\mu_{\mathcal{A}}^d : \text{hom}_{\mathcal{A}}(L_{d-1}, L_d) \otimes \dots \otimes \text{hom}_{\mathcal{A}}(L_0, L_1) \rightarrow \text{hom}_{\mathcal{A}}(L_0, L_1). \quad (2.1.1)$$

These must satisfy relations, usually called the A_∞ associativity equations

$$\sum_{n=r+s+t, s \geq 1} \mu_{\mathcal{A}}^{r+t+1}(1^{\otimes r} \otimes \mu_{\mathcal{A}}^s \otimes 1^{\otimes t}) = 0 \quad (2.1.2)$$

for each n . When there is no ambiguity about which category we are working in, we will suppress the subscript \mathcal{A} on both hom and μ^d . For a given \mathcal{A} , the cohomology category $H(\mathcal{A})$ has the same objects, and morphisms $\text{Hom}(L_0, L_1) := H(\text{hom}(L_0, L_1), \mu^1)$. This inherits a product from μ^2 , defined by

$$\text{Hom}(L_1, L_2) \otimes \text{Hom}(L_0, L_1) \rightarrow \text{Hom}(L_0, L_2) \quad (2.1.3)$$

$$([b], [a]) \mapsto b \cdot a := [\mu^2(b, a)] \quad (2.1.4)$$

for any two cocycles $a \in \text{hom}(L_0, L_1)$, $b \in \text{hom}(L_1, L_2)$. By a ‘representative’ of an A_∞ category we will mean any quasi-isomorphic category.

An A_∞ category might additionally have one, or more, of the following features:

- finiteness: the morphism spaces are finite-dimensional;

- cohomological finiteness: the morphism spaces of $H(\mathcal{A})$ are finite-dimensional;
- minimality: μ^1 vanishes;
- c-unitality: the cohomology category $H(\mathcal{A})$ is unital;
- strictly unitality: for each $L \in \text{Ob}\mathcal{A}$, there is an element $1_L \in \text{hom}(L, L)$ such that $\mu^1(1_L) = 0$, $\mu^2(a, 1_L) = \mu^2(1'_L, a) = a$ for all $a \in \text{hom}(L, L')$, and higher compositions involving 1_L are zero.

2.1.2 Choosing ‘nice’ representatives for A_∞ categories

Given any A_∞ category \mathcal{A} , we can always construct a minimal quasi-isomorphic A_∞ category with the same objects. If \mathcal{A} is also c-unital, minimality and strict unitality can be achieved simultaneously ([62, Lemma 2.1]). Also, if two strictly unital A_∞ algebras are quasi-isomorphic, and the quasi-isomorphism is c-unital, then the algebras can be related by a quasi-isomorphism that is itself strictly unital ([45, Theorem 3.2.2.1]).

Fix any minimal, strictly unital A_∞ algebra B .

Lemma 2.1.1. *Suppose \mathcal{A} is a c-unital A_∞ category with a distinguished object L such that the A_∞ algebra $\text{hom}_{\mathcal{A}}(L, L)$ is quasi-isomorphic to B , and that the quasi-isomorphism is c-unital. Then we can find a quasi-isomorphic category $\tilde{\mathcal{A}}$, with the same objects, that is minimal, strictly unital, and such that $\text{hom}_{\tilde{\mathcal{A}}}(L, L)$ is strictly isomorphic to B .*

Proof. First fix an A_∞ category \mathcal{A}' that is minimal, strictly unital, and quasi-isomorphic to \mathcal{A} , with the same objects. The A_∞ algebra $\text{hom}_{\mathcal{A}'}(L, L)$ is strictly unital, and quasi-isomorphic to the strictly unital algebra B . We can find a quasi-isomorphism between $\text{hom}_{\mathcal{A}'}(L, L)$ and B that is strictly unital. Say it is given by maps

$$f_n : \text{hom}_{\mathcal{A}'}(L, L)^{\otimes n} \rightarrow B. \quad (2.1.5)$$

As \mathcal{A}' and B are minimal, f_1 is just an automorphism, and the f_n can be extended to a formal diffeomorphism (as defined in [62, Section 1]) on \mathcal{A}' . Moreover, we can require that the maps

$$\text{hom}_{\mathcal{A}'}(X_{d-1}, X_d) \otimes \dots \otimes \text{hom}_{\mathcal{A}'}(X_0, X_1) \rightarrow \text{hom}_{\mathcal{A}'}(X_0, X_1) \quad (2.1.6)$$

giving the formal diffeomorphism vanish whenever $d \geq 2$, $X_i = X_{i+1}$ for some i , and the element of $\text{hom}_{\mathcal{A}'}(X_i, X_{i+1})$ we are plugging in is the strict unit. Call $\tilde{\mathcal{A}}$ the A_∞ category produced by the formal diffeomorphism; it is easy to check that our requirement forces $\tilde{\mathcal{A}}$ to be strictly unital, and that the units are the images of the units of \mathcal{A}' . Thus $\tilde{\mathcal{A}}$ is a suitable category. \square

Remark 2.1.2. *If \mathcal{A} is finite or cohomologically finite, and (where applicable) B is finite dimensional, then it is immediate that the quasi-isomorphic categories described above are also finite.*

2.1.3 Background on twisted complexes

Let \mathcal{A} be a finite, strictly unital A_∞ category. We summarize the material we will use on twisted complexes in \mathcal{A} , as introduced by [43]. If \mathcal{A} were not unital, one would need far more

care; although most computations are routine and omitted, we have tried to flag all times that strict unitality is used.

Additive enlargement. The additive enlargement of \mathcal{A} , $\Sigma\mathcal{A}$, is an A_∞ category whose objects are formal direct sums

$$X = \bigoplus_{i \in I} V^i \otimes X^i \quad (2.1.7)$$

where I is a finite set, $\{X^i\}_{i \in I}$ a family of objects of \mathcal{A} , and $\{V^i\}_{i \in I}$ a family of finite dimensional \mathbb{Z}_2 vector spaces. The morphisms between two of these objects are made up of morphisms between the constituent summands, tensored with the spaces of linear maps between the vector spaces: for instance,

$$\text{hom}_{\Sigma\mathcal{A}}(V^0 \otimes X^0, V^1 \otimes X^1) = \text{hom}_{\mathbb{Z}_2}(V^0, V^1) \otimes \text{hom}_{\mathcal{A}}(X^0, X^1). \quad (2.1.8)$$

The A_∞ compositions maps are inherited from those of \mathcal{A} , combined with usual composition of linear maps. We will often denote both vector and A_∞ morphism spaces by hom . The identity endomorphism of a vector space V will be denoted Id_V or Id , whereas the strict unit of $\text{hom}(L, L)$ will be denoted by 1_L or 1 .

Twisted complexes. We will work with the category of twisted complexes in \mathcal{A} , $Tw\mathcal{A}$. Objects consist of pairs (X, δ_X) , where $X \in \text{Ob}\Sigma\mathcal{A}$, and the connection (or differential) δ_X is an element of $\text{hom}_{\Sigma\mathcal{A}}(X, X)$ such that

- there is a finite, decreasing filtration by subcomplexes

$$X = F^0 X \supset F^1 X \supset \dots \supset F^n X = 0 \quad (2.1.9)$$

such that the induced connection on the quotients $F^i X / F^{i+1} X$ is zero. (Subcomplexes consist of objects $\bigoplus_{i \in I} W^i \otimes X^i$, where each W^i is a vector subspace of V^i , that are preserved by the connection δ_X .)

- the connection satisfies the generalised Maurer-Cartan equation

$$\sum_{r=1}^{\infty} \mu_{\Sigma\mathcal{A}}^r(\delta_X, \dots, \delta_X) = 0. \quad (2.1.10)$$

The morphism spaces are the same as for $\Sigma\mathcal{A}$; hereafter the subscript will be dropped. A key difference with $\Sigma\mathcal{A}$ is that all the compositions are now deformed by contributions from the connections:

$$\mu_{Tw\mathcal{A}}^d(a_d, \dots, a_1) = \sum_{i_0, \dots, i_d} \mu_{\Sigma\mathcal{A}}^{d+i_0+\dots+i_d}(\overbrace{\delta_{X_d}, \dots, \delta_{X_d}}^{i_d}, a_d, \overbrace{\delta_{X_{d-1}}, \dots, \delta_{X_{d-1}}}^{i_{d-1}}, a_{d-1}, \dots, a_1, \overbrace{\delta_{X_0}, \dots, \delta_{X_0}}^{i_0}) \quad (2.1.11)$$

where the sum is over all non-negative integers i_k . This makes $Tw\mathcal{A}$ into an A_∞ category. We will denote the associated cohomology category by $H(Tw\mathcal{A})$, and its morphism groups by $\text{Hom}_{Tw\mathcal{A}}$. \mathcal{A} embeds into $Tw\mathcal{A}$ as a subcategory: each object $X \in \mathcal{A}$ gives an object

$(X, \delta_X = 0)$ of $Tw\mathcal{A}$. If \mathcal{A} is finite, $Tw\mathcal{A}$ clearly is too. Moreover, strict unitality of \mathcal{A} implies strict unitality of $Tw\mathcal{A}$. $H(Tw\mathcal{A})$ is a triangulated category (see [32]). notice that the shift functor $[1]$ is the identity. Notably, this means there is a collection of distinguished triangles; for each such triangle, say

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \swarrow h \quad \searrow g & \\ & C & \end{array} \quad (2.1.12)$$

and any $Z \in \text{Ob}Tw\mathcal{A}$, there are exact sequences

$$\dots \rightarrow \text{Hom}_{Tw\mathcal{A}}(Z, A) \rightarrow \text{Hom}_{Tw\mathcal{A}}(Z, B) \rightarrow \text{Hom}_{Tw\mathcal{A}}(Z, C) \rightarrow \dots \quad (2.1.13)$$

and

$$\dots \rightarrow \text{Hom}_{Tw\mathcal{A}}(B, Z) \rightarrow \text{Hom}_{Tw\mathcal{A}}(A, Z) \rightarrow \text{Hom}_{Tw\mathcal{A}}(C, Z) \rightarrow \dots \quad (2.1.14)$$

in the cohomology category, where the maps are pre- and post-composition with f , g and h .

Quotients and cones. Given a twisted complex (X, δ_X) and a subcomplex $(Y, \delta_X|_Y)$, one can form the quotient complex by taking vector space quotients piece-wise; this inherits a connection from X . We denote this by $(X/Y, \delta_{X/Y})$. Let $\pi : X \rightarrow X/Y$ be the quotient map, and $i : Y \rightarrow X$ the inclusion. These are cochains (this uses strict unitality), so make sense as a morphisms in the cohomology category $H(Tw\mathcal{A})$. Moreover, there is a distinguished triangle

$$\begin{array}{ccc} X/Y & \xrightarrow{\quad} & Y \\ & \swarrow \pi \quad \searrow i & \\ & X & \end{array} \quad (2.1.15)$$

Consider a morphism between twisted complexes, say

$$c : X \rightarrow Y. \quad (2.1.16)$$

Whenever $\mu_{Tw\mathcal{A}}^1(c) = 0 \in \text{hom}(X, Y)$, we can complete this to a distinguished triangle as follows: define the cone of c , itself an element of $Tw\mathcal{A}$:

$$\text{Cone}(c) = X \oplus Y \quad (2.1.17)$$

with

$$\delta_{\text{Cone}(c)} = \begin{pmatrix} \delta_X & 0 \\ c & \delta_Y \end{pmatrix} \quad (2.1.18)$$

As we are working with \mathbb{Z}_2 coefficients and no gradings, there is not need for the shifts or negative signs that the reader might be used to. Together with c , the obvious maps $\text{Cone}(c) \rightarrow X$ (projection) and $Y \rightarrow \text{Cone}(c)$ (inclusion) fit into a distinguished triangle.

Lemma 2.1.3. *Suppose (X, δ_X) is a twisted complex, $(Y, \delta_X|_Y)$ a subcomplex, and that Y sits in a distinguished triangle with $A \xrightarrow{f} B$, such that f is an isomorphism. Then Y is acyclic, and $\pi : X \rightarrow X/Y$ is an isomorphism in $H(Tw(\mathcal{A}))$.*

Proof. Consider the distinguished triangle

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow & \swarrow \\ & Y & \end{array} \quad (2.1.19)$$

For each $Z \in \text{Ob} Tw(\mathcal{A})$ there is an exact sequence

$$\dots \rightarrow \text{Hom}_{Tw\mathcal{A}}(Z, A) \xrightarrow{f_*} \text{Hom}_{Tw\mathcal{A}}(Z, B) \rightarrow \text{Hom}_{Tw\mathcal{A}}(Z, Y) \rightarrow \dots \quad (2.1.20)$$

and by hypothesis, post-composition f_* is an isomorphism; thus, by Yoneda's lemma, Y is isomorphic to 0 in $H(Tw(\mathcal{A}))$. Considering the distinguished triangle

$$\begin{array}{ccc} X/Y & \xrightarrow{\quad} & Y \\ & \searrow & \swarrow \\ & X & \end{array} \quad (2.1.21)$$

another use of Yoneda's lemma gives the desired conclusion. \square

Evaluation maps. For any pair $X, Y \in \text{Ob} Tw\mathcal{A}$, as $\text{hom}(X, Y)$ is finite, $\text{hom}(X, Y) \otimes X$ is an element of $\Sigma\mathcal{A}$; moreover, setting

$$\delta_{\text{hom}(X, Y) \otimes X} = \mu_{Tw\mathcal{A}}^1 \otimes 1 + Id \otimes \delta_X$$

gives it the structure of a twisted complex. (This uses strict unitality; see e.g. [62], section (3o).) There are canonical isomorphisms

$$\begin{aligned} \text{hom}(\text{hom}(X, Y) \otimes X, Y) &\cong \text{hom}_{\mathbb{Z}_2}(\text{hom}(X, Y), \mathbb{Z}_2) \otimes \text{hom}(X, Y) \\ &\cong \text{hom}(X, Y)^\vee \otimes \text{hom}(X, Y) \cong \text{End}_{\mathbb{Z}_2}(\text{hom}(X, Y)). \end{aligned} \quad (2.1.22)$$

The evaluation map

$$ev : \text{hom}(X, Y) \otimes X \longrightarrow Y \quad (2.1.23)$$

corresponds to the identity in $\text{End}_{\mathbb{Z}_2}(\text{hom}(X, Y))$: for any \mathbb{Z}_2 -vector space basis for $\text{hom}(X, Y)$, say e_i , we have

$$ev = \sum e_i^\vee \otimes e_i \quad (2.1.24)$$

One can check that $\mu_{Tw\mathcal{A}}^1(ev) = 0$.

(This too uses strict unitality of $Tw\mathcal{A}$.) Thus we can talk about its cone, the twisted complex $\text{Cone}(ev)$. We shall later use the following observation:

Lemma 2.1.4. *Let X_i be objects of $\Sigma\mathcal{A}$, for $i = 0, \dots, r$. We denote by ev_i the evaluation map*

$$\text{hom}(X_i, X_{i+1}) \otimes X_i \rightarrow X_{i+1} \quad (2.1.25)$$

and let

$$Z_i := \text{hom}(X_{r-1}, X_r) \otimes \dots \otimes \text{hom}(X_i, X_{i+1}) \otimes X_i. \quad (2.1.26)$$

Tensoring with the identity, ev_i extends to a map $Z_i \rightarrow Z_{i+1}$, that we also denote by ev_i . Fix $W \in \text{Ob}\Sigma\mathcal{A}$, and let $a_r \otimes \dots \otimes a_{i+1} \otimes b$ denote an element of

$$\text{hom}(W, Z_i) = \text{hom}(X_{r-1}, X_r) \otimes \dots \otimes \text{hom}(X_i, X_{i+1}) \otimes \text{hom}(W, X_i). \quad (2.1.27)$$

Then

$$\mu_{\Sigma\mathcal{A}}^{r-i+1}(ev_r, ev_{r-1}, \dots, ev_i, a_r \otimes a_{r-1} \otimes \dots \otimes a_{i+1} \otimes b) = \mu_{\Sigma\mathcal{A}}^{r-i+1}(a_r, a_{r-1}, \dots, a_{i+1}, b) \in \text{hom}(W, X_r). \quad (2.1.28)$$

Proof. This is nothing more than an exercise in definitions. For instance, in the simplest case, we have $a \in \text{hom}(X_0, X_1)$, $b \in \text{hom}(W, X_0)$, and

$$ev : \text{hom}(X_0, X_1) \otimes X_0 \rightarrow X_1. \quad (2.1.29)$$

Write $ev = \sum e_i^y \otimes e_i \in \text{hom}(X_0, X_1)^\vee \otimes \text{hom}(X_0, X_1)$, for some basis e_i of $\text{hom}(X_0, X_1)$, and $a = \sum a_i \cdot e_i$, some scalars a_i . We have

$$\mu_{\Sigma\mathcal{A}}^2(ev, a \otimes b) = \mu_{\Sigma\mathcal{A}}^2\left(\sum e_i^y \otimes e_i, a \otimes b\right) \quad (2.1.30)$$

$$= \sum \mu_{\Sigma\mathcal{A}}^2(e_i^y \otimes e_i, a \otimes b) \quad (2.1.31)$$

$$= \sum (e_i^y \circ a) \mu_{\mathcal{A}}^2(e_i, b) \quad (2.1.32)$$

$$= \sum a_i \mu_{\mathcal{A}}^2(e_i, b) \quad (2.1.33)$$

$$= \mu_{\mathcal{A}}^2\left(\sum a_i \cdot e_i, b\right) \quad (2.1.34)$$

$$= \mu_{\mathcal{A}}^2(a, b). \quad (2.1.35)$$

□

2.2 The Fukaya category of an exact manifold: preliminaries

Let (M^{2n}, ω, θ) be an exact symplectic manifold of dimension $2n > 2$ with contact type boundary, and J an ω -compatible almost complex structure of contact type near the boundary. Unless otherwise specified, all Lagrangians will hereafter be assumed to be exact, compact, and disjoint from the boundary. Note Lagrangian spheres are automatically exact.

We will use the Fukaya category $\mathcal{Fuk}(M)$ of M , as introduced in [26]; we follow the exposition of [62, Section 9]. $\mathcal{Fuk}(M)$ is an A_∞ category with objects the Lagrangians of M , and morphisms Floer chain groups between Lagrangians. Our Floer chain complexes have \mathbb{Z}_2 coefficients, and there are no gradings. We start by giving a rapid overview of the set-up, calling attention to some of the features we shall use (subsections 2.2.1 and 2.2.2). We then describe the A_∞ algebra structure of $CF(L, L)$, for a Lagrangian sphere (subsection 2.2.3). Finally, subsection 2.2.4 gives some criteria for two Lagrangian spheres to be quasi-isomorphic as objects of the Fukaya category.

2.2.1 Floer cohomology

Fix Lagrangians L_0 and L_1 . Let $\mathcal{H} = C_c^\infty(\text{Int}(M), \mathbb{R})$, and let \mathcal{J} be the space of all ω -compatible almost complex structures that agree with J near the boundary of M .

A *Floer datum* for the pair (L_0, L_1) consists of $J_{L_0, L_1} \in C^\infty([0, 1], \mathcal{J})$ and $H_{L_0, L_1} \in C^\infty([0, 1], \mathcal{H})$ such that $\phi^1(L_0) \pitchfork L_1$, where ϕ is the flow of the time-dependent Hamiltonian vector field X corresponding to H . Let

$$\mathcal{C}(L_0, L_1) = \{y : [0, 1] \rightarrow M \mid y(0) \in L_0, y(1) \in L_1, dy/dt = X(t, y(t))\}. \quad (2.2.1)$$

Its elements are naturally in bijection with the set $\phi^1(L_0) \pitchfork L_1$.

Given a Floer datum, the Floer cochain group $CF(L_0, L_1)$ is the \mathbb{Z}_2 vector space generated by the elements of $\mathcal{C}(L_0, L_1)$. One equips it with a differential, μ^1 , that counts certain ‘pseudo-holomorphic strips’ as follows: fix $y_0, y_1 \in \mathcal{C}(L_0, L_1)$. Let Z be the Riemann surface $\mathbb{R} \times [0, 1] \subset \mathbb{C}$, with coordinates (s, t) . $\mathcal{M}_Z(y_0, y_1)$ is the set of maps $u \in C^\infty(Z, M)$ satisfying:

- Floer’s equation: $\partial_s u + J(t, u(s, t))(\partial_t u - X(t, u)) = 0$
- boundary conditions: $u(s, 0) \in L_0, u(s, 1) \in L_1$
- asymptotic conditions: $\lim_{s \rightarrow +\infty} u(s, \cdot) = y_1, \lim_{s \rightarrow -\infty} u(s, \cdot) = y_0$.

There is an \mathbb{R} -action on $\mathcal{M}_Z(y_0, y_1)$: translation in the s -variable. For $y_0 \neq y_1$, this action is free; let $\mathcal{M}_Z^*(y_0, y_1)$ be the quotient space. For $y_0 = y_1$, $\mathcal{M}_Z(y_0, y_1)$ is a point, because of exactness, and we set $\mathcal{M}_Z^*(y_0, y_1) = \emptyset$.

Additionally, we put a constraint on our Floer datum: we require that the space $\mathcal{M}_Z(y_0, y_1)$ defined above be *regular*; this is a property that implies that it is smooth and of the appropriate, expected dimension, given by the index of a certain surjective Fredholm operator. This condition is called transversality; see e.g. [25, 52]. It is satisfied for a generic choice of Floer datum. Note that we are free to choose any H_{L_0, L_1} such that $\phi^1(L_0) \pitchfork L_1$, as long as we don’t want to exercise any additional control on J_{L_0, L_1} .

Given a regular Floer datum, the boundary operator on $CF(L_0, L_1)$, called the Floer differential, is given by

$$\mu^1(y_1) = \sum_{y_0} \#\mathcal{M}_Z^*(y_0, y_1) \cdot y_0 \quad (2.2.2)$$

where $\#$ counts the number of isolated points mod 2.

Properties. The boundary operator μ^1 is a differential. The associated cohomology,

$$H(CF(L_0, L_1), \mu^1) =: HF(L_0, L_1) \quad (2.2.3)$$

is called the Lagrangian Floer cohomology of L_0 and L_1 .

$HF(L_0, L_1)$ is independent of auxiliary choices, up to canonical isomorphism. It is independent of the choice of θ away from ∂M : we could have chosen instead $\theta + df$, for any $f \in C^\infty(M, \mathbb{R})$ supported away from ∂M . Also, it is finite as a \mathbb{Z}_2 vector space. It is invariant under exact Lagrangian isotopy of L_0 or L_1 .

Remark 2.2.1. *Whenever the Floer cochain group can be graded by Maslov indices, as in e.g. [57], μ^1 increases the degrees by one. This is why we talk of Floer cochains and cohomology.*

For an two Lagrangians L_0 and L_1 , we will denote by $hf(L_0, L_1)$ the dimension of $HF(L_0, L_1)$ as a \mathbb{Z}_2 vector space.

Duality. Suppose $(H_{L_0, L_1}, J_{L_0, L_1})$ is a regular Floer datum for the pair (L_0, L_1) . Then a regular Floer datum for (L_1, L_0) is given by

$$H_{L_1, L_0}(t) = -H_{L_0, L_1}(1-t) \quad J_{L_1, L_0}(t) = J_{L_0, L_1}(1-t). \quad (2.2.4)$$

We will refer to this as the ‘dual Floer datum’ for (L_1, L_0) . Given these choices, there is a one-to-one correspondance between generators of $CF(L_0, L_1)$ and $CF(L_1, L_0)$, and between the Floer discs counted by their differentials μ^1 . These identify $CF(L_1, L_0)$ with the dual complex of $CF(L_0, L_1)$; we will call corresponding generators ‘dual’ to one another. One gets a vector space isomorphism $HF(L_0, L_1) \cong (HF(L_1, L_0))^v$.

2.2.2 The Fukaya category

Suppose you have already chosen a regular Floer datum $(H_{L_0, L_1}, J_{L_0, L_1})$ for each pair of Lagrangians (L_0, L_1) . The Fukaya category has objects the Lagrangians of M , and morphisms Floer chain groups between Lagrangians. We want to define maps

$$\mu^d : CF(L_{d-1}, L_d) \otimes CF(L_{d-2}, L_{d-1}) \otimes \dots \otimes CF(L_0, L_1) \rightarrow CF(L_0, L_d) \quad (2.2.5)$$

for $d \geq 2$, that, together with the Floer differential, should satisfy the A_∞ associativity equations for each n . Roughly speaking, the μ^d are obtained by counting certain ‘pseudo-holomorphic’ discs with $d+1$ boundary marked points. Here are a few more details.

Fix d . Let \mathcal{D} be a closed unit complex disc with some $d+1$ boundary points removed; label these ζ_0, \dots, ζ_d , ordered anti-clockwise. ζ_0 will be known as an incoming point, and ζ_i ($i \geq 1$) as outgoing ones. Consider the half-infinite holomorphic strips $\mathbb{R}^\pm \times [0, 1] \subset \mathbb{C}$. Equip \mathcal{D} with *strip-like ends*: proper holomorphic embeddings

$$\epsilon_0 : \mathbb{R}^- \times [0, 1] \rightarrow \mathcal{D} \quad \text{and} \quad \epsilon_j : \mathbb{R}^+ \times [0, 1] \rightarrow \mathcal{D} \quad \text{for } j = 1, \dots, d$$

with disjoint images, such that $\epsilon_j^{-1}(\partial\mathcal{D}) = \mathbb{R}^\pm \times \{0, 1\}$ and $\lim_{s \rightarrow \pm\infty} \epsilon_j(s, \cdot) = \zeta_j$ for $j = 0, \dots, d$.

The boundary $\partial\mathcal{D}$ consists of $d+1$ connected components; let C_i be the segment between ζ_i and ζ_{i+1} , with indices taken modulo $d+1$. *Lagrangian labels* for \mathcal{D} are the assignment of a Lagrangian, say L_i , to each C_i .

A *perturbation datum* for \mathcal{D} is a pair (K, J) where $J \in C^\infty(\mathcal{D}, \mathcal{J})$ and $K \in \Omega^1(\mathcal{D}, \mathcal{H})$ such that $K(\xi)|_{L_j} = 0$ for all $\xi \in TC_j \subset T(\partial\mathcal{D})$, any j . We require it to be compatible with choice of strip-like ends and the Floer data, in the following sense:

$$\epsilon_{\zeta_j}^* K = H_{\zeta_j}(t) dt, \quad J(\epsilon_{\zeta_j}(s, t)) = J_{\zeta_j}(t) \quad (2.2.6)$$

for all j , and all $(s, t) \in \mathbb{R}^\pm \times [0, 1]$.

Let $Y \in \Omega^1(\mathcal{D}, C^\infty(TM))$ be the Hamiltonian vector field valued one-form associated to K . Given $a_i \in CF(L_{i-1}, L_1)$, the naive approach would be to define $\mu^d(a_d, \dots, a_1)$ by counting solutions $u \in C^\infty(\mathcal{D}, M)$ to the generalized Floer equation

$$Du(z) + J(z, u) \circ Du(z) \circ i = Y(z, u) + J(z, u) \circ Y(z, u) \circ i \quad (2.2.7)$$

with boundary conditions given by the Lagrangian labels L_i and asymptotic conditions given by the a_i , and to sum over all possible \mathcal{D} (allowing the marked points to move). As with Floer's equation, one would additionally need the perturbation data to be regular.

The problem is that for the A_∞ relations to be satisfied, the auxiliary data needs to be chosen far more carefully. There are various solutions to this, one of which ([62]) uses strip-like ends and perturbation data defined on classifying spaces for families of holomorphic discs with $d+1$ marked points, called ‘universal’ choices. It is shown that these choices can be made to be ‘consistent’: roughly, this enables one to carry through the requisite gluing arguments. Analogously to the discussion before, the consistent universal choice of perturbation data is required to be compatible with the Floer data and universal choice of strip-like ends. These choices are generically regular.

The construction of such strip-like ends and perturbation data starts with any choice of regular Floer data, and inducts on d . We note the following feature of this induction: for a fixed d , and suppose we are looking to equip discs with $d+1$ marked points with adequate perturbation data. For a given collection of Lagrangian labels, the choice of perturbation datum made in the induction only depends on the data for discs with $k+1$ marked points, $k < d$, and Lagrangian labels an ordered subset of our collection.

Properties. This gives an A_∞ category. Different admissible choices of auxiliary data (strip-like ends, Floer and perturbation data) give a quasi-isomorphic A_∞ category; the quasi-isomorphism fixes the objects. Working instead with $(M, \omega, \theta + df)$, for some $f \in C^\infty(M, \mathbb{R})$ with support away from ∂M , gives an isomorphic A_∞ category. Thus the product that the cohomology category $H(\mathcal{Fuk}(M))$ inherits from μ^2 is independent of all these choices. Moreover, with this structure $H(\mathcal{Fuk}(M))$ is unital (it is a linear category), which means that $\mathcal{Fuk}(M)$ is c-unital. In particular, it makes sense to talk about quasi-isomorphic objects (see section 2.2.4); notice that any two Lagrangians that are isotopic through exact Lagrangians are quasi-isomorphic as objects of $\mathcal{Fuk}(M)$.

2.2.3 A_∞ algebra associated to a Lagrangian sphere.

Zero-section of a cotangent bundle. Let L be a sphere of dimension at least 2. T^*L has a standard exact symplectic structure; denote the usual one-form by α , and by $\omega_L = d\alpha$ the symplectic form. The zero-section is an exact Lagrangian; we will simply denote it by L . Fix a metric on L ; for any $\delta > 0$, the closed disc bundle of radius δ , say B_δ , is an exact symplectic manifold with contact type boundary. We can find an ω -compatible almost complex structure j , of contact type near the boundary.

Lemma 2.2.2. *Let A be the A_∞ algebra on two generators given by $\mathbb{Z}_2[\epsilon]/(\epsilon^2)$, with $\mu^2(1, \epsilon) = \mu^2(\epsilon, 1) = \epsilon$, and all other A_∞ structure maps trivial. $CF_{B_\delta}(L, L)$ is quasi-isomorphic to A .*

Many variations or partial forms of this statement exist in the literature, going back to [28]. One might think of this fact as a consequence of two features of Floer cohomology:

- **PSS isomorphism.** $HF(L, L)$ is isomorphic, as a ring, to the usual (e.g. simplicial or singular) cohomology ring $H^*(L; \mathbb{Z}_2)$. The maps between them are usually called ‘PSS isomorphisms’. For Lagrangian Floer theory, these were constructed by Albers [6], following work of Piunikhin, Salamon and Schwarz [54] for Hamiltonian Floer theory.

Albers does not discuss the ring structure, but one can check that [54]’s arguments carry over – see [39].

- **Gradings.** $c_1(T^*L) = 0$, and the zero section L has Maslov class zero. $CF(L, L)$ carries a canonical \mathbb{Z} grading ([27, Theorem 2.3]), giving it the structure of a *graded* A_∞ algebra.

To prove Lemma 2.2.2, it would be enough to check that the PSS isomorphisms are compatible with these gradings. Why? $CF(L, L)$ is certainly isomorphic to A as a differential algebra, by PSS. Take a minimal, strictly unital model for $CF(L, L)$; it has a distinguished pair of generators, which, abusing notation slightly, we denote by 1 and ϵ . Compatibility with gradings would mean that 1 has grading 0 , and ϵ has grading $\dim(L) \geq 2$. Together with strict unitality, these imply that all higher A_∞ products must be zero.

Recent work of Abouzaid [1] includes a detailed proof of Lemma 2.2.2 (using a different strategy), so we do not discuss the above further.

Recall that when defining $CF(L, L)$, we are free to choose the Hamiltonian perturbation for the pair (L, L) . It will sometimes be useful to make the following choice: fix a Morse function on L with two critical points, say h . Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a bump function centered at 0 , with support on $[-\delta/2, \delta/2]$. Let H be the function on B_δ given by $H(q, p) = h(q)\psi(\|p\|)$, where $q \in L$, $p \in T_q^*L$, and $\|\cdot\|$ is our choice of metric. Set $H_{L,L}(t) = H$, for all $t \in [0, 1]$, to be the Hamiltonian for (L, L) . Provided h is sufficiently small, its critical points are by construction in one-to-one correspondence with generators of $CF(L, L)$. We assume this to be the case. Let x_M be the critical point corresponding to the maximum, and x_m the one corresponding to the minimum. One can check from the Morse cohomology gradings, and e.g. Abouzaid’s proof, that on the level of cohomology, x_M corresponds to 1 , and x_m to ϵ . (In our later discussion it will only actually matter that one critical point can be identified with 1 and the other with ϵ .) We shall make use of this in the proof of Proposition 2.2.7.

Arbitrary exact Lagrangian sphere. Let $L \subset M$ be a Lagrangian sphere. We claim that

Proposition 2.2.3. *$CF(L, L)$ is isomorphic to A as an A_∞ algebra.*

As before, let B_δ denote the disc bundle of radius δ of T^*L . By Weinstein, there exists $\delta > 0$ and an embedding $\iota : B_\delta \hookrightarrow M$ such that the zero section gets mapped to L , and $\iota^*\omega = \omega_L$. By exactness, $\iota^*\theta = \alpha + dg$, some smooth function g . Any Floer or (universal) perturbation datum for B_δ can be extended to one on M :

- Extend all Hamiltonian perturbations by zero.
- j induces an ω -compatible almost complex structure on $\iota(B_\delta)$. As we are only using structures on B_δ that agree with j on a neighbourhood of ∂B_δ , it is enough to extend this. First extend j to an ω_L -compatible almost complex structure on $B_{2\delta}$. This gives an ω -compatible almost complex structure on $\iota(B_{2\delta})$. Using the fact that the space of ω -compatible almost complex structures on $\iota(B_{2\delta} \setminus B_\delta)$ is contractible, we can find j' , an ω -compatible almost complex structure on M , such that j' restricts to j on a neighbourhood of $\iota(B_\delta)$, and $j' = J$ outside $\iota(B_{2\delta})$.

Make the same choice of universal strip-like ends as for B_δ .

Lemma 2.2.4. (Abouzaid, see e.g. [2] or [62, Lemma 7.5]) *Suppose (N, ω_N, θ_N) is an arbitrary exact symplectic manifold, (U, ω_U, θ_U) an exact symplectic manifold of the same dimension with contact type boundary, and $\iota : U \hookrightarrow \text{int}(N)$ an embedding such that $\iota^* \omega_N = \omega_U$, $\iota^* \theta_N = \theta_U + dg$. Suppose we also have an ω_N -compatible almost complex structure \mathbb{J} , whose restriction to U is of contact type near the boundary.*

Let S be a compact connected Riemann surface with boundary, and $u : S \rightarrow N$ a \mathbb{J} -holomorphic map such that $u(\partial S) \subset \text{int}(U)$. Then $u(S) \subset \text{int}(U)$ as well. Moreover, this only requires u to be \mathbb{J} -holomorphic in a neighbourhood of ∂U . (In particular, there could be a compactly supported perturbation on the interior of U .)

This lemma implies that no pseudo-holomorphic disc corresponding to the extended data can leave $\iota(B_\delta)$. Thus regularity of the data for B_δ implies regularity of its extension. The observation at the end of section 2.2.2 implies that this extended data can be taken as part of consistent universal choices made to define $\mathcal{Fuk}(M)$. Suppose you have set-up the auxiliary data for $\mathcal{Fuk}(M)$ in this way. Using lemma 2.2.4, we see that

$$CF_{B_\delta}(L, L) = CF_M(L, L) \quad (2.2.8)$$

as A_∞ algebras, ‘on the nose’. Proposition 2.2.3 now follows from lemma 2.2.2.

2.2.4 Isomorphism in the Fukaya category: some criteria

$\mathcal{Fuk}(M)$ is c-unital, so there is a meaningful notion of quasi-isomorphism between its objects: two Lagrangians L_0, L_1 are quasi-isomorphic in $\mathcal{Fuk}(M)$ if there are μ^1 -closed morphisms $a \in CF(L_0, L_1)$, $b \in CF(L_1, L_0)$ such that

$$[\mu^2(b, a)] = 1_{L_0} \in HF(L_0, L_0) \quad \text{and} \quad [\mu^2(a, b)] = 1_{L_1} \in HF(L_1, L_1) \quad (2.2.9)$$

We will call two such objects ‘Fukaya isomorphic’ or ‘quasi-isomorphic in the Fukaya category’. When some of the objects are spheres, seemingly weaker conditions turn out to be equivalent to this one.

Lemma 2.2.5. *Let L_0 be a Lagrangian sphere, and L_1 any Lagrangian. Suppose that we have $a \in CF(L_0, L_1)$, $b \in CF(L_1, L_0)$ such that*

$$[\mu^2(a, b)] = 1_{L_1} \in HF(L_1, L_1) \quad (2.2.10)$$

Then L_0 and L_1 are Fukaya isomorphic objects.

Proof. $(b \cdot a)^2 = b \cdot a$, so $b \cdot a$ is an idempotent element of $HF(L_0, L_0)$. Moreover, as $a \cdot b \cdot a = a \cdot b \cdot a$ is non-zero. As we know the ring structure of $HF(L_0, L_0)$, we can check that the only other idempotent is 1_{L_0} itself, which means that $b \cdot a = 1_{L_0}$. \square

Corollary 2.2.6. *Suppose that L_0 and L_1 are Lagrangian spheres, and that multiplication*

$$HF(L_0, L_1) \otimes HF(L_1, L_0) \rightarrow HF(L_1, L_1) \quad (2.2.11)$$

is surjective. Then L_0 and L_1 are Fukaya isomorphic.

Proof. It is enough to find elements $a \in HF(L_0, L_1)$ and $b \in HF(L_0, L_1)$ such that $a \cdot b = 1_{L_1} \in HF(L_1, L_1)$. Consider the PSS ring isomorphism $HF(L_1, L_1) \cong H^*(L_1; \mathbb{Z}_2)$. We know that the image of multiplication $HF(L_0, L_1) \otimes HF(L_1, L_0) \rightarrow H^*(L_1; \mathbb{Z}_2)$ cannot be contained in $H^n(L_1)$. As all elements in $H^*(L_1; \mathbb{Z}_2) \setminus H^n(L_1; \mathbb{Z}_2)$ are invertible, there are $a \in HF(L_0, L_1)$ and $c \in HF(L_1, L_0)$ such that $a \cdot c$ is invertible in $HF(L_1, L_1)$. Let $d \in HF(L_1, L_1)$ be the inverse. Notice that by construction, we have $a \cdot (c \cdot d) = 1_{L_1}$. \square

Proposition 2.2.7. *Suppose that L_0 is a Lagrangian sphere, and that the map*

$$HF(L_0, L_1) \rightarrow HF(L_0, L_1) \tag{2.2.12}$$

$$a \mapsto a \cdot \epsilon \tag{2.2.13}$$

is non-zero. Then there exists $d \in HF(L_1, L_0)$ such that $d \cdot a = 1_{L_0}$.

Proof. Fix regular Floer data for the pairs (L_0, L_1) and (L_0, L_0) , say $(H_{L_0, L_1}, J_{L_0, L_1})$ and $(H_{L_0, L_0}, J_{L_0, L_0})$. For convenience, assume H_{L_0, L_0} is constructed from a Morse function on L_0 , as described in section 2.2.3. By hypothesis, there must be cocycles $a \in CF(L_0, L_1)$ and $c \in CF(L_0, L_1)$, non zero in cohomology, such that $\mu^2(a, x_m) = c$. We can assume that a corresponds to a unique point of $L_0 \pitchfork \phi^1(L_1)$, where ϕ^1 is the time-one flow associated to H_{L_0, L_1} .

Let e_i be generators of $CF_{\bar{F}}(L_0, L_1)$, each corresponding to a point of $L_0 \pitchfork \phi^1(L_1)$. Up to holomorphic reparametrization, there is a unique closed disc \mathcal{D} in \mathbb{C} with three boundary points removed. W.l.o.g. it is the unit disc with the roots of unity removed. Label these anti-clockwise as ζ_0 (incoming), ζ_1 and ζ_2 (both outgoing). Suppose you have chosen the auxiliary data to define a Fukaya category \mathcal{F} of M , using $(H_{L_0, L_1}, J_{L_0, L_1})$ and $(H_{L_0, L_0}, J_{L_0, L_0})$. \mathcal{D} has been equipped with strip-like ends, and a perturbation datum for each choice of Lagrangian labels. The coefficient of e_i in the product $\mu^2(a, x_m)$ is computed by counting certain solutions $u \in C^\infty(\mathcal{D}, M)$ to the generalized Floer equation corresponding to the perturbation datum, and boundary and asymptotic conditions determined by figure 2-1.

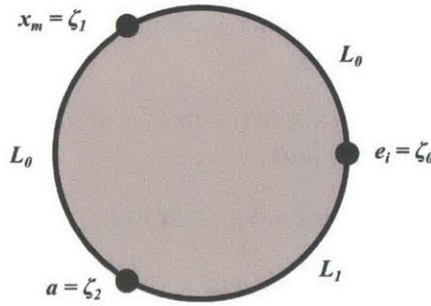


Figure 2-1: Lagrangian labels and asymptotic conditions for marked points of \mathcal{D} .

We now want to study the product $HF(L_1, L_0) \otimes HF(L_0, L_1) \rightarrow HF(L_0, L_0)$. Define this at the chain level by making the following choices: use the same Floer datum for (L_0, L_1) as before, the dual datum for (L_1, L_0) , and, for (L_0, L_0) , the datum *dual* to the original datum $(H_{L_0, L_0}, J_{L_0, L_0})$. With our new choice of datum, the two generators for $CF(L_0, L_0)$ are still x_M and x_m , but now x_M corresponds to ϵ (for it is the minimum of $-h$), and x_m to 1_{L_0} .

Let \mathcal{D}' be another copy of the closed unit disc in \mathbb{C} with the third roots of unity removed, labelled anticlockwise by ζ'_0 , ζ'_1 and ζ'_2 . As with \mathcal{D} , ζ'_0 is considered an incoming point, and ζ'_1 and ζ'_2 outgoing ones. Equip \mathcal{D}' with the Lagrangian labels as in figure 2-2.

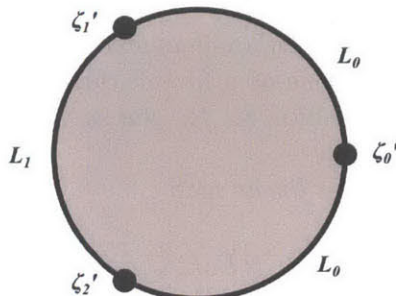


Figure 2-2: Lagrangian labels for \mathcal{D}' .

Rotation by $2\pi/3$ gives a map $\mathcal{D}' \rightarrow \mathcal{D}$ such that $\zeta'_i \mapsto \zeta_{i+1}$, with indices taken mod 3. Post- and pre-composing with rotation give strip-like ends and a perturbation datum for \mathcal{D}' , using those for \mathcal{D} (when equipped with Lagrangian labels as in figure 2-1). We have chosen our Floer data precisely for these to be compatible with it. Let $\tilde{\mathcal{F}}$ be a Fukaya category defined using these and additional admissible auxiliary data.

Denote by e_i^y the generator of $CF_{\tilde{\mathcal{F}}}(L_1, L_0)$ dual to e_i . The ‘obvious’ linear pairing defined by

$$CF_{\tilde{\mathcal{F}}}(L_1, L_0) \otimes CF_{\tilde{\mathcal{F}}}(L_0, L_1) \rightarrow \mathbb{Z}_2 \quad (2.2.14)$$

$$e_i^y \otimes e_i \rightarrow 1 \quad (2.2.15)$$

descends to a non-degenerate pairing on cohomology, say (\cdot, \cdot) . Say $c = c_1 + \dots + c_k$, where each c_i corresponds to one point of $L_0 \pitchfork \phi^1(L_1)$. Let c_i^y be the generator of $CF_{\tilde{\mathcal{F}}}(L_1, L_0)$ dual to c_i . Pick a class $d \in HF(L_1, L_0)$ such that $(d, c) = 1$. Note that in general, d need not be unique. Fix a chain level representative for d , which we also denote by d . It must be of the form

$$d = c_{k_1}^y + \dots + c_{k_l}^y + f_1^y + \dots + f_j^y \quad (2.2.16)$$

where $\{k_1, \dots, k_l\} \subset \{1, \dots, k\}$ and l is *odd*, and each f_i is a chain element corresponding to a point of $L_0 \pitchfork \phi^1(L_1)$ disjoint from the c_i , with f_i^y denoting its dual.

We want to compute $\mu^2(d, a)$. By construction, the maps $u \in C^\infty(\mathcal{D}, M)$ counted in the products $CF_{\mathcal{F}}(L_0, L_1) \otimes CF_{\mathcal{F}}(L_0, L_0) \rightarrow CF_{\mathcal{F}}(L_0, L_1)$ are in one-to-one correspondance with the maps $v \in C^\infty(\mathcal{D}, M)$ counted in the products $CF_{\tilde{\mathcal{F}}}(L_1, L_0) \otimes CF_{\tilde{\mathcal{F}}}(L_0, L_1) \rightarrow C_{\tilde{\mathcal{F}}}(L_0, L_0)$. Suppose the asymptotics conditions for u are those of figure 2-1. Then v will satisfy asymptotic conditions as follows:

- ζ'_0 , an incoming point, corresponds to x_m , which has cohomology class 1_{L_0} .
- ζ'_1 , which is an outgoing point, as is ζ_2 , corresponds to a .
- ζ'_2 is an outgoing point, whereas ζ_0 is an incoming one; this means that ζ'_2 corresponds to e_i^y .

The mod 2 count of these discs is one precisely when e_i is one of the c_i . Thus, as l is odd, the coefficient of 1_{L_0} in $\mu^2(d, a)$ is one. This means that either $d \cdot a = 1$ or $d \cdot a = 1 + \epsilon$. If the former holds, we are done; if the later does, simply note that $(d + \epsilon \cdot d) \cdot a = 1 + \epsilon + \epsilon \cdot (1 + \epsilon) = 1$. This completes the proof of Proposition 2.2.7. \square

Remark 2.2.8. *Although we took a different approach for technical reasons, the informed reader might want to think of the above as a formal consequence of the Frobenius property of the Fukaya category (see e.g. [62, Section 8c] for statements), together with the ring structure of $HF(L_0, L_0)$. Here is an outline:*

For any Lagrangian L , we have a linear map

$$\int_L : HF(L, L) \rightarrow \mathbb{Z}_2 \quad (2.2.17)$$

which under the PSS isomorphism corresponds to integration over the fundamental class. Moreover, for any two Lagrangians L_0 and L_1 , the pairing

$$HL(L_1, L_0) \otimes HF(L_0, L_1) \xrightarrow{\mu^2} HF(L_0, L_0) \xrightarrow{\int_{L_0}} \mathbb{Z}_2 \quad (2.2.18)$$

is non-degenerate (Frobenius property). One could check that this is the same pairing as the one we used in the proof. In this case, knowing that it factors through the Floer product readily provides an element $d \in HF(L_1, L_0)$ such that $d \cdot c \neq 0 \in HF(L_0, L_0)$. By associativity of Floer products, we have

$$(d \cdot a) \cdot \epsilon \neq 0 \in HF(L_0, L_0). \quad (2.2.19)$$

L_0 is a sphere; in particular, we completely understand the ring structure of $HF(L_0, L_0)$. Considering all possibilities, we see that $d \cdot a = 1$ or $d \cdot a = 1 + \epsilon$. Now conclude as before.

Corollary 2.2.9. *Suppose L_0, L_1 are Lagrangians, with L_0 a sphere, and*

$$HF(L_0, L_1) \rightarrow HF(L_0, L_1) \quad (2.2.20)$$

$$[a] \mapsto [\mu_2(a, \epsilon)] \quad (2.2.21)$$

has non-trivial image. Then L_0 and L_1 are Fukaya isomorphic.

Proof. This is immediate from Proposition 2.2.7 and Lemma 2.2.5. \square

2.3 Classification of finite dimensional A -modules

Let us start by recalling some definitions [41, Section 4.2].

Definition 2.3.1. *Given an A_∞ algebra B , a right A_∞ module over B is a \mathbb{Z}_2 vector space M together with a sequence of maps*

$$\mu_M^n : M \otimes B^{\otimes(n-1)} \rightarrow M \quad (2.3.1)$$

for all $n \geq 1$, such that

$$\sum \mu_M^{r+t+1}(Id^r \otimes \mu^s \otimes Id^t) = 0 \quad (2.3.2)$$

where the sum is taken over all decompositions $r + s + t = n$, with $r, t \geq 0$ and $s > 0$, and μ^s denotes μ_B^s when $r > 0$, and μ_M^s otherwise.

If B is strictly unital, and μ_M^n vanishes whenever one of the entries from B is its unit, we say that M is strictly unital.

Definition 2.3.2. A morphism of right A_∞ -modules $f : N \rightarrow M$ is a sequence of morphisms

$$f_n : N \otimes A^{\otimes(n-1)} \rightarrow M \quad (2.3.3)$$

for $n \geq 1$ such that for all n , we have

$$\sum f_{r+1+t}(1^{\otimes r} \otimes \mu^s \otimes 1^{\otimes t}) = \sum \mu_{j+1}(f_l \otimes 1^{\otimes j}) \quad (2.3.4)$$

where the left hand sum is taken over all decompositions $n = r + s + t$, with $r, t \geq 0$, $s \geq 1$, and the right-hand sum is taken over all decompositions $n = j + l$, $j \geq 0$, with $l \geq 1$. The morphism is said to be strictly unital if f_n ($n \geq 2$) vanishes whenever one of the entries for A is the unit 1.

The definitions for left modules are analogous.

Recall A is the strictly unital A_∞ algebra $\mathbb{Z}_2[\epsilon]/\epsilon^2$ with the following A_∞ structure: $\mu^2(1, \epsilon) = \mu^2(\epsilon, 1) = \epsilon$, and all other maps are zero. We wish to classify strictly unital, finite dimensional A_∞ modules over A . For such a right A_∞ module, say M , the only (potentially) non-trivial action of A is given by $\mu^1(\cdot)$, $\mu^2(\cdot, \epsilon)$, $\mu^3(\cdot, \epsilon, \epsilon)$, \dots and similarly for left modules. For $k \geq 1$, define R_k to be the right A -module with two generators (as a vector space), r_k^0 and r_k^1 , such that $\mu^k(r_k^0, \epsilon, \dots, \epsilon) = r_k^1$, and the A_∞ action is trivial otherwise. Define \mathcal{L}_k , a left A -module with generators l_k^0 and l_k^1 , similarly.

Proposition 2.3.3. Let M be a strictly unital, finite dimensional right (resp. left) A_∞ module over A . Then M is quasi-isomorphic to a finite dimensional module N , that decomposes as a direct sum of A_∞ modules of the following forms:

- copies of \mathbb{Z}_2 , with the trivial A_∞ action;
- finitely many R_k 's (resp. \mathcal{L}_k 's).

We present a proof for right-modules; as A is isomorphic to its opposite algebra, the case of left-modules follows. Let t be a formal variable; define a differential on $M[[t]]$ by setting

$$d(a) = \mu^1(a) + t\mu^2(a, \epsilon) + \dots + t^n \mu^{n+1}(a, \epsilon, \dots, \epsilon) + \dots \quad (2.3.5)$$

for all $a \in M$, and extending this linearly in t to $M[[t]]$. The A_∞ relations for M imply that $d^2 = 0$. Consider $H(M[[t]], d)$; as M is finite (as a \mathbb{Z}_2 -vector space), this is a finitely generated $\mathbb{Z}_2[[t]]$ -module. Since $\mathbb{Z}_2[[t]]$ is a P.I.D., the standard decomposition theorem gives:

$$H(M[[t]], d) \cong \mathbb{Z}_2[[t]] \oplus \dots \oplus \mathbb{Z}_2[[t]] \oplus \mathbb{Z}_2[[t]]/(t^{k_1}) \oplus \dots \oplus \mathbb{Z}_2[[t]]/(t^{k_n}) \quad (2.3.6)$$

for some positive integers k_i .

Our strategy is as follows: using this description, we construct a finite dimensional A_∞ module N , with structure as described in the proposition, together with a quasi-isomorphism

$$(N[[t]], d) \rightarrow (M[[t]], d). \quad (2.3.7)$$

We then check that the map this induces between N and M is a quasi-isomorphism of A_∞ modules.

Let r be the number of free summands in the direct sum decomposition of $H(M[[t]], d)$. We define N to be an A -module with $r + 2n$ generators (as a \mathbb{Z}_2 -vector space), given by the direct sum of the following A -modules: r copies of \mathbb{Z}_2 , and R_{k_i} for $i = 1, \dots, n$. Now consider $N[[t]]$, with differential d defined as in equation (2.3.5). By construction, d is zero on the first r copies of $\mathbb{Z}_2[[t]]$, $d(r_{k_i}^0) = t^{k_i} r_{k_i}^1$, and $d(r_{k_i}^1) = 0$. There is an obvious isomorphism of $\mathbb{Z}_2[[t]]$ -modules $\tilde{f} : H(N[[t]], d) \cong H(M[[t]], d)$. $N[[t]]$ is a free $\mathbb{Z}_2[[t]]$ -module; we lift \tilde{f} to a quasi-isomorphism $f : N[[t]] \rightarrow M[[t]]$ of $\mathbb{Z}_2[[t]]$ -modules, by specifying the image of our given basis:

- Let e_i , $1 \leq i \leq r$, be a generator for the i^{th} $\mathbb{Z}_2[[t]]$ summand in $N[[t]]$. This is naturally a generator for the i^{th} $\mathbb{Z}_2[[t]]$ summand of $H(N[[t]], d)$; map it to any lift in $M[[t]]$ of its image under \tilde{f} .
- Map $r_{k_i}^1$ to any lift in $M[[t]]$ of the generator of $\mathbb{Z}_2[[t]]/(t^{k_i})$, say ρ_i^1 . Then $t^{k_i} \rho_i^1 = 0 \in H(M[[t]], d)$; pick ρ_i^0 such that $t^{k_i} \rho_i^1 = d\rho_i^0$ in $M[[t]]$. Now set $f(r_{k_i}^0) = \rho_i^0$.

This gives a well-defined chain map, that, by construction, is an isomorphism on cohomology.

Let $\Pi_n : M[[t]] \rightarrow M$ be the projection to the coefficient of t^n in the power series. We define the maps f_n by setting

$$f_n(a, \overbrace{\epsilon, \dots, \epsilon}^{n-1}) = \Pi_{n-1}(f(a))$$

and requiring f to be strictly unital. We must check that this is indeed a map of A_∞ modules. This essentially follows from the fact that $f : N[[t]] \rightarrow M[[t]]$ was a chain map, though one has to pay attention to some details. Let us spell these out.

Fix n , and consider $f_{n+1} : N \otimes A^{\otimes n} \rightarrow M$. First suppose you set all entries for A to ϵ . Note that for a summand on the left-hand side of equation (2.3.4) to be non-zero, we need $r = 0$. (At this point the A_∞ structure on $\text{hom}(L, L)$ is crucial: knowing the multiplicative structure for cohomology does not suffice.) As f is a chain map, for all $a \in N$, $df(a) = fd(a) \in M[[t]]$. Taking Π_n , this precisely tells us that equation (2.3.4) is satisfied. It remains to consider the case where some of the entries for A are set to 1. From strict unitality, equation (2.3.4) holds trivially unless there is exactly one entry equal to 1, and all others are ϵ . Let us examine the left-hand side of equation (2.3.4). If 1 is in the final position, the only (potential) non-zero terms are

- for $r = 0$, $f_1(\mu_2(a, 1)) = f_1(a)$ for $n = 2$;
- for $r \neq 0$, $f_l(a, \epsilon, \dots, \epsilon, \mu_2(\epsilon, 1)) = f_l(a, \epsilon, \dots, \epsilon)$ for $n = l + 1 \geq 3$.

If 1 is not in the final position, there are exactly two non-zero terms, both for $s = 2$. One involves $\mu_2(1, \epsilon)$, and the other $\mu_2(\epsilon, 1)$ or $\mu_2(a, 1)$, and, after contracting, they cancel out.

The right-hand side of equation (2.3.4) can only be non-zero when 1 is in the final position, in which case we get

$$\mu_2(f_l(a, \epsilon, \dots, \epsilon), 1) = f_l(a, \epsilon, \dots, \epsilon) \quad (2.3.8)$$

for $n = l + 1 \geq 2$. Thus the map we defined is indeed a morphism of A_∞ -modules.

Finally, we show that $f : N \rightarrow M$ is a quasi-isomorphism. Consider the two short exact sequences of chain complexes, fitting into a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & N[[t]] & \xrightarrow{t} & N[[t]] & \longrightarrow & N \longrightarrow 0 \\ & & \downarrow f & & \downarrow f & & \downarrow f_1 \\ 0 & \longrightarrow & M[[t]] & \xrightarrow{t} & M[[t]] & \longrightarrow & M \longrightarrow 0 \end{array} \quad (2.3.9)$$

Each short exact sequence induces a long exact sequence on cohomology, and the vertical maps in the diagram induce maps between these long exact sequences. As we already know that f is a quasi-isomorphism, the five-lemma tells us that f_1 is one, too.

2.4 Twisted complexes and iterations of Dehn twists

Suppose M is an exact symplectic manifold with contact type boundary, L a Lagrangian sphere, and L_0 any exact Lagrangian. This section contains the computation of an expression for the iterated Dehn twist $\tau_L^n L_0$ as a twisted complex. The starting point is:

Theorem 2.4.1. ([61, Theorem 1] and [62, Corollary 17.17]) *Suppose A is a representative of the Fukaya category of M that is finite and strictly unital. Then $\tau_L L_0$, the Dehn twist of L_0 about L , is quasi-isomorphic to*

$$\text{Cone}(ev : \text{hom}(L, L_0) \otimes L \rightarrow L_0) \quad (2.4.1)$$

as objects of $\text{Tw}\mathcal{A}$.

A basic property of twisted complexes, that gets us off the ground, is:

Lemma 2.4.2. *Suppose X and Y are quasi-isomorphic in $\text{Tw}\mathcal{A}$; let Z be any object of $\text{Tw}\mathcal{A}$, and*

$$ev_X : \text{hom}_{\text{Tw}\mathcal{A}}(Z, X) \otimes Z \rightarrow X \quad (2.4.2)$$

$$ev_Y : \text{hom}_{\text{Tw}\mathcal{A}}(Z, Y) \otimes Z \rightarrow Y \quad (2.4.3)$$

Then $\text{Cone}(ev_X)$ and $\text{Cone}(ev_Y)$ are quasi-isomorphic objects of $\text{Tw}\mathcal{A}$.

Proof. This follows immediately from [62], Corollary 3.16. □

This means that one can write down an expression for $\tau_L^n L_0$ as an iteration of cones, and look to simplify it.

We know that \mathcal{A} is cohomologically finite and c -unital (subsection 2.2.2), and that $CF_{\mathcal{A}}(L, L)$ is quasi-isomorphic to A (Proposition 2.2.3). By Proposition 2.1.1, we may assume that \mathcal{A} is

minimal, finite, strictly unital, and that $\text{hom}_{\mathcal{A}}(L, L) = A$ as A_{∞} algebras. We will heavily rely on the the simplicity of this structure to reduce the expression for $\tau_L^n L_0$.

The manipulations of twisted complexes that follow are valid for any finite, strictly unital A_{∞} category \mathcal{A} , with a distinguished element L such that $\text{hom}(L, L) = A$, and L_0, L_1 any objects of \mathcal{A} . Unlike the previous assumptions, it is not crucial that \mathcal{A} be minimal; it will simply make our computations, and the complexes we encounter, cleaner.

Proposition 2.4.3. *Let $T_L^n L_0$ denote the twisted complex*

$$L_0 \oplus \text{hom}(L, L_0) \otimes L \oplus \text{hom}(L, L_0) \otimes (\epsilon) \otimes L \oplus \dots \oplus \text{hom}(L, L_0) \otimes \overbrace{(\epsilon) \dots (\epsilon)}^{n-1} \otimes L \quad (2.4.4)$$

where the connection is given by

- 0 on the first summand;
- ev , the evaluation map $\text{hom}(L, L_0) \otimes L \rightarrow L_0$ on the second summand;
- $Id \otimes ev + \mu^2 \otimes 1$ on the third summand, where ev is the evaluation map $\epsilon \otimes L \rightarrow L$;
- ...
- On the r^{th} summand,

$$Id^{\otimes(r-2)} \otimes ev + \sum (\mu^i \otimes Id^{\otimes j}) \otimes 1 \quad (2.4.5)$$

where the sum ranges over all decompositions $r - 1 = i + j$, with $i > 1$. Note that $\mu^i \otimes Id^{\otimes j} \otimes 1$ maps to the $(j + 2)^{\text{th}}$ summand, and $Id^{\otimes(r-2)} \otimes ev$ maps to the $(r - 1)^{\text{th}}$ summand.

We claim that $T_L^n L_0$ is quasi-isomorphic to $\tau_L^n L_0$.

Remark 2.4.4. (ϵ) is a one-dimensional vector space; one could also present $T_L^n L_0$ using multiple copies of $\text{hom}(L, L_0) \otimes L$ and push the ϵ 's over to the expression for the connection. We will obtain the complex in Proposition as a quotient; from that prospective, the presentation above seemed the most natural one.

We shall prove this by induction on n . Theorem 2.4.1 gives the case $n = 1$. We start with a preliminary computation.

Lemma 2.4.5. *Fix an object L_1 of $\mathcal{A} \subset Tw\mathcal{A}$. Then*

$$\begin{aligned} \text{hom}(L_1, T_L^n L_0) &= \text{hom}(L_1, L_0) \oplus \text{hom}(L, L_0) \otimes \text{hom}(L_1, L) \\ &\oplus \text{hom}(L, L_0) \otimes (\epsilon) \otimes \text{hom}(L_1, L) \oplus \dots \oplus \text{hom}(L, L_0) \otimes (\epsilon) \dots (\epsilon) \otimes \text{hom}(L_1, L) \end{aligned} \quad (2.4.6)$$

and $\mu_{Tw\mathcal{A}}^1$ acts

- on the first summand, by zero;
- on the second summand, by $\mu_{\mathcal{A}}^2$;

• ...

• on the r^{th} summand, by

$$\sum_{r=i+j, j>1} (Id^{\otimes i} \otimes \mu_{\mathcal{A}}^j + \mu_{\mathcal{A}}^j \otimes Id^{\otimes i}). \quad (2.4.7)$$

Proof. By definition,

$$\mu_{Tw\mathcal{A}}^1(a) = \sum_{i \geq 0} \mu_{\Sigma\mathcal{A}}^{1+i}(\delta_{T_L^n L_0}, \dots, \delta_{T_L^n L_0}, a) \quad (2.4.8)$$

After expanding, the claim boils down to the A_∞ -associativity equations for $hom(L, L_0)$ together with Lemma 2.1.4. Let us do the calculation for the third summand. Consider a class $c = a \otimes \epsilon \otimes b$ where $a \in hom(L, L_0)$ and $b \in hom(L_1, L)$. We have

$$\begin{aligned} \mu_{Tw\mathcal{A}}^1(a \otimes \epsilon \otimes b) &= \mu_{\Sigma\mathcal{A}}^1(a \otimes \epsilon \otimes b) + \mu_{\Sigma\mathcal{A}}^2(Id \otimes ev + \mu^2 \otimes 1, a \otimes \epsilon \otimes b) \\ &\quad + \mu_{\Sigma\mathcal{A}}^3(ev, Id \otimes ev + \mu^2 \otimes 1, a \otimes \epsilon \otimes b) \end{aligned} \quad (2.4.9)$$

$$\begin{aligned} &= 0 + \mu_{\Sigma\mathcal{A}}^2(Id \otimes ev, a \otimes \epsilon \otimes b) + \mu_{\Sigma\mathcal{A}}^2(\mu^2 \otimes 1, a \otimes \epsilon \otimes b) \\ &\quad + \mu_{\Sigma\mathcal{A}}^3(ev, \mu^2 \otimes 1, a \otimes \epsilon \otimes b) + \mu_{\Sigma\mathcal{A}}^3(ev, Id \otimes ev, a \otimes \epsilon \otimes b) \end{aligned} \quad (2.4.10)$$

$$= a \otimes \mu^2(\epsilon, b) + \mu^2(a, \epsilon) \otimes b + 0 + \mu^3(a, \epsilon, b) \quad (2.4.11)$$

where we made use of the minimality of the A_∞ structure of \mathcal{A} at the second step, and of its strict unitality at the third. \square

Suppose the claim in Proposition 2.4.4 is true for all powers of τ_L up to τ_L^{n-1} . By Theorem 2.4.1 and Lemma 2.4.2

$$\tau_L^n(L_0) \cong Cone(ev : hom(L, T_L^{n-1}L_0) \otimes L \rightarrow T_L^{n-1}L_0) \quad (2.4.12)$$

$$= T_L^{n-1}L_0 \oplus hom(L, T_L^{n-1}L_0) \otimes L \quad (2.4.13)$$

with the cone connection.

Lemma 2.4.5 provides us with an expression for the differential on $hom(L, T_L^{n-1}L_0)$, and thus for the cone connection on

$$T_L^{n-1}L_0 \oplus hom(L, T_L^{n-1}L_0) \otimes L \quad (2.4.14)$$

solely in terms of the composition maps of \mathcal{A} . Let us denote this complex by $\widehat{T}_L^n L_0$. Our strategy is as follows: by repeated quotienting of subcomplexes, we exhibit a complex that is quasi-isomorphic to $\widehat{T}_L^n L_0$, and of the desired form. Each subcomplex we shall quotient out by will be the direct sum of two pieces between which the connection gives an isomorphism, ensuring that quotienting does not change what element of $H(Tw\mathcal{A})$ we have (see Lemma 2.1.3). This is chiefly a matter of book-keeping; we start by presenting the cases $n = 2$ and 3 , in the hope that they render the general computation less forbidding.

The twisted complex $\widehat{T}_L^2 L_0$ is given by

$$\begin{array}{ccc}
\text{hom}(L, L_0) \otimes \text{hom}(L, L) \otimes L & \xrightarrow{\text{Id} \otimes \text{ev}} & \text{hom}(L, L_0) \otimes L \\
\downarrow \mu^2 \otimes 1 & & \downarrow \text{ev} \\
\text{hom}(L, L_0) \otimes L & \xrightarrow{\text{ev}} & L_0
\end{array} \tag{2.4.15}$$

where the complex is the direct sum of the four terms in the diagram, and the maps give the connection δ (we will suppress subscripts on connections for the rest of this section). $T_L^{n-1}L_0$ is on the right column, and $\text{hom}(L, T_L^{n-1}L_0) \otimes L$ on the left one. Consider $\text{hom}(L, L_0) \otimes 1 \otimes L$; its image under δ is the diagonal copy Δ of $\text{hom}(L, L_0) \otimes L$; moreover, $\text{hom}(L, L_0) \otimes 1 \otimes L$, together with its image, form a subcomplex. Quotient this out to get the complex

$$\text{hom}(L, L_0) \otimes \epsilon \otimes L \xrightarrow{\mu^2 \otimes 1 + \text{Id} \otimes \text{ev}} \text{hom}(L, L_0) \otimes L \xrightarrow{\text{ev}} L_0 \tag{2.4.16}$$

Further, note that δ gave an isomorphism

$$\text{hom}(L, L_0) \otimes 1 \otimes L \xrightarrow{(\mu^1 \otimes 1, \text{Id} \otimes \text{ev})} \Delta \subset \text{hom}(L, L_0) \otimes L \times \text{hom}(L, L_0) \otimes L \tag{2.4.17}$$

Thus the new complex is quasi-isomorphic to the old one, using Lemma 2.1.3.

Can this be generalised? Let us look at $n = 3$. We get:

$$\begin{array}{ccc}
\text{hom}(L, L_0) \otimes \epsilon \otimes \text{hom}(L, L) \otimes L & \xrightarrow{\text{Id} \otimes \text{ev}} & \text{hom}(L, L_0) \otimes \epsilon \otimes L \\
\downarrow \mu^2 \otimes \text{Id} \otimes 1 + \text{Id} \otimes \mu^2 \otimes 1 & & \downarrow \mu^2 \otimes 1 + \text{Id} \otimes \text{ev} \\
\mu^3 \otimes 1 \left(\begin{array}{ccc} \text{hom}(L, L_0) \otimes \text{hom}(L, L) \otimes L & \xrightarrow{\text{Id} \otimes \text{ev}} & \text{hom}(L, L_0) \otimes L \\ \downarrow \mu^2 \otimes 1 & & \downarrow \text{ev} \\ \text{hom}(L, L_0) \otimes L & \xrightarrow{\text{ev}} & L_0 \end{array} \right. & &
\end{array} \tag{2.4.18}$$

As before, the image of $\text{hom}(L, L_0) \otimes 1 \otimes L$ under δ is the diagonal copy of $\text{hom}(L, L_0) \otimes L$, and

- $\text{hom}(L, L_0) \otimes 1 \otimes L$ and its image under δ form a subcomplex.
- δ gives an isomorphism of $\text{hom}(L, L_0) \otimes 1 \otimes L$ onto its image.

The quotient complex, quasi-isomorphic to the old one, is:

$$\begin{array}{ccc}
\text{hom}(L, L_0) \otimes \epsilon \otimes \text{hom}(L, L) \otimes L & \xrightarrow{\text{Id} \otimes \text{ev}} & \text{hom}(L, L_0) \otimes \epsilon \otimes L \\
\downarrow \mu^2 \otimes \text{Id} \otimes 1 + \text{Id} \otimes \mu^2 \otimes 1 & \searrow \mu^3 \otimes 1 & \downarrow \mu^2 \otimes 1 + \text{Id} \otimes \text{ev} \\
\text{hom}(L, L_0) \otimes \epsilon \otimes L & \xrightarrow{\mu^2 \otimes 1 + \text{Id} \otimes \text{ev}} & \text{hom}(L, L_0) \otimes L \\
& & \downarrow \text{ev} \\
& & L_0
\end{array} \tag{2.4.19}$$

Now, consider the image under δ of $\text{hom}(L, L_0) \otimes \epsilon \otimes 1 \otimes L$; this is the diagonal copy of $\text{hom}(L, L_0) \otimes \epsilon \otimes L$. ($\mu^2 \otimes \text{Id} \otimes 1$ has no image by construction, and $\mu^3 \otimes 1$ has none because of unitality.) Again, $\text{hom}(L, L_0) \otimes \epsilon \otimes 1 \otimes L$ and its image form a subcomplex, and δ gives an isomorphism between them. Quotienting gives the quasi-isomorphic twisted complex

$$\begin{array}{ccccc} \text{hom}(L, L_0) \otimes \epsilon \otimes \epsilon \otimes L & \xrightarrow{\mu^2 \otimes \text{Id} \otimes 1 + \text{Id} \otimes \mu^2 \otimes \text{ev}} & \text{hom}(L, L_0) \otimes \epsilon \otimes L & \xrightarrow{\mu^2 \otimes 1 + \text{Id} \otimes \text{ev}} & \text{hom}(L, L_0) \otimes L \xrightarrow{\text{ev}} L \\ & \searrow & & \nearrow & \\ & & & & \mu^3 \otimes 1 \end{array} \quad (2.4.20)$$

which we recognise as $T_L^3 L_0$. (As $\mu^2(\epsilon, \epsilon) = 0$, the $\text{Id} \otimes \mu^2 \otimes 1$ term has vanished. See also remark 2.4.6.)

For a general n , proceed similarly:

- First quotient out $\text{hom}(L, L_0) \otimes 1 \otimes L$ and its image under δ , the diagonal copy of $\text{hom}(L, L_0) \otimes L$; the new complex has one copy of L_0 , one of $\text{hom}(L, L_0) \otimes L$, two of $\text{hom}(L, L_0) \otimes \epsilon \otimes L$, and terms with higher tensor length.
- Then quotient out $\text{hom}(L, L_0) \otimes \epsilon \otimes 1 \otimes L$ and its image under δ , which is the diagonal copy of $\text{hom}(L, L_0) \otimes \epsilon \otimes L$; the new complex agrees with $T_L^n L_0$ for terms containing up to two tensor signs, has two copies of $\text{hom}(L, L_0) \otimes \epsilon \otimes \epsilon \otimes L$, and terms with a higher number of tensor signs.

• ...

- At the r^{th} stage, we quotient out $\text{hom}(L, L_0) \otimes \overbrace{\epsilon \otimes \dots \otimes \epsilon}^{r-1} \otimes 1 \otimes L$, and its image under δ . Notice that until this point, $\text{hom}(L, L_0) \otimes \overbrace{\epsilon \otimes \dots \otimes \epsilon}^{r-1} \otimes 1 \otimes L$ was unaffected by the quotienting process.

• ...

- Finally, quotient out $\text{hom}(L, L_0) \otimes \overbrace{\epsilon \otimes \dots \otimes \epsilon}^{n-2} \otimes 1 \otimes L$ and its image under δ , the diagonal copy of $\text{hom}(L, L_0) \otimes \overbrace{\epsilon \otimes \dots \otimes \epsilon}^{n-2} \otimes L$. This gives the complex $T_L^n L_0$.

This completes the proof of Proposition 2.4.4.

Remark 2.4.6. *Suppose that we started with weaker hypothesis: instead of having $\text{hom}(L, L) = A$, we only had $\text{hom}(L, L) = B$, where $B = \mathbb{Z}_2(\epsilon)/(\epsilon^2)$ is a minimal, strictly unital A_∞ algebra with $\mu^2(1, \epsilon) = \mu(\epsilon, 1) = \epsilon$, but some of the higher compositions might be non-zero. (The other assumptions on \mathcal{A} are unchanged.) Then one could check that $\tau_E^n L_0$ is quasi-isomorphic to*

$$L_0 \oplus \text{hom}(L, L_0) \otimes L \oplus \text{hom}(L, L_0) \otimes \epsilon \otimes L \oplus \dots \oplus \text{hom}(L, L_0) \otimes \overbrace{\epsilon \dots \epsilon}^{n-1} \otimes L \quad (2.4.21)$$

with connection acting on the r^{th} summand by

$$\text{Id}^{\otimes r-2} \otimes \text{ev} \oplus \sum_{r=i+j+k, j>1} \text{Id}^{\otimes i} \otimes \mu_{\mathcal{A}}^j \otimes \text{Id}^{\otimes k-1} \otimes 1. \quad (2.4.22)$$

2.5 Lower bounds on $hf(\tau_L^n L_0, L_1)$

2.5.1 $T_L^n L_0$ as a cone

The twisted complex $T_L^n L_0$ of Proposition 2.4.4 can itself be viewed as a cone, as follows.

Definition 2.5.1. Let $C_L^n L_0$ be the twisted complex

$$\begin{aligned} \text{hom}(L, L_0) \otimes L \oplus \text{hom}(L, L_0) \otimes \epsilon \otimes L \oplus \text{hom}(L, L_0) \otimes \epsilon \otimes \epsilon \otimes L \oplus \dots \\ \oplus \text{hom}(L, L_0) \otimes \epsilon \otimes \dots \otimes \epsilon \otimes L \end{aligned} \quad (2.5.1)$$

with connection

$$Id^{k-2} \otimes ev + \sum_{k-1=i+j, i>1} (\mu^i \otimes Id^j) \otimes 1 \quad (2.5.2)$$

on the k^{th} summand, $k > 0$. (This is a quotient of $T_L^n L_0$.)

There is a map of twisted complexes $C_L^n L_0 \rightarrow L_0$ given by $ev : \text{hom}(L, L_0) \otimes L \rightarrow L_0$ on the first summand, and zero elsewhere; by construction, $T_L^n L_0$ is the cone of this map.

Lemma 2.5.2. Suppose $L_1 \in \text{Ob}A \subset \text{Ob}Tw(A)$. Then

$$\begin{aligned} \text{hom}(L_1, C_L^n L_0) = \text{hom}(L, L_0) \otimes \text{hom}(L_1, L) \oplus \text{hom}(L, L_0) \otimes \epsilon \otimes \text{hom}(L_1, L) \\ \oplus \text{hom}(L, L_0) \otimes \epsilon \otimes \epsilon \otimes \text{hom}(L_1, L) \oplus \dots \oplus \text{hom}(L, L_0) \otimes \epsilon \otimes \dots \otimes \epsilon \otimes \text{hom}(L_1, L) \end{aligned} \quad (2.5.3)$$

and μ_{TwA}^1 acts on the k^{th} summand by

$$\sum_{i+j=k+1, i,j>0} (\mu^i \otimes Id^j + Id^i \otimes \mu^j) \quad (2.5.4)$$

Proof. This is essentially the same computation as for Lemma 2.4.5. \square

Notice this is the reduced bar complex for the tensor $\text{hom}(L, L_0) \otimes_A \text{hom}(L_1, L)$, truncated (see e.g. [45, Section 2.3.3]).

2.5.2 Quasi-isomorphic truncated bar complexes

Let M be a right A -module, and N a left A -module; assume moreover that they are both finite and strictly unital. We will use the following notation for the truncated reduced bar complex of their tensor product

$$(M \otimes_A N)_n := M \otimes N \oplus M \otimes \epsilon \otimes N \oplus \dots \oplus M \otimes \underbrace{\epsilon \otimes \dots \otimes \epsilon}_{n-1} \otimes N \quad (2.5.5)$$

with the natural differential induced by the module structures on M and N : on the r^{th} summand, it is

$$\sum_{r-1=i+j, i,j>0} (\mu^i \otimes Id^{\otimes j} + Id^{\otimes j} \otimes \mu^i) \quad (2.5.6)$$

($\mu_{T_w A}^1$ on $\text{hom}(L_1, C_L^n L_0)$, in Lemma 2.5.2, is an example of this.) Let $H(M \otimes_A N)_n$ denote the cohomology of this chain complex. We shall use the following:

Lemma 2.5.3. *Let M' (resp. N') be a finite, strictly unital right (resp. left) A -module. Suppose that M' is quasi-isomorphic to M , and N' quasi-isomorphic to N . Then there is a quasi-isomorphism*

$$h : (M' \otimes_A N')_n \rightarrow (M \otimes_A N)_n \quad (2.5.7)$$

Proof. Suppose $f : M' \rightarrow M$ is a quasi-isomorphism of left A -modules, given by a collection of maps

$$f_r : M' \otimes A^{\otimes(r-1)} \rightarrow M \quad (2.5.8)$$

for all $r \geq 1$. By Lefèvre-Hasegawa ([45, Theorem 3.2.2.1]), we may assume f is a strictly unital map. Let $\tilde{f} : (M' \otimes_A N')_n \rightarrow (M \otimes_A N)_n$ be the ‘obvious’ map constructed using all of the f_r : \tilde{f} acts on

$$M' \otimes \underbrace{\epsilon \otimes \epsilon \dots \otimes \epsilon}_k \otimes N' \quad (2.5.9)$$

by

$$\tilde{f}(a \otimes \epsilon \otimes \dots \otimes \epsilon \otimes b) = \sum_{i=1}^{k+1} f_i(a, \epsilon, \dots, \epsilon) \otimes \underbrace{\epsilon \otimes \dots \otimes \epsilon}_{k-i+1} \otimes b. \quad (2.5.10)$$

\tilde{f} is a chain map (this follows from the equations satisfied by f as an A -module morphism). Similarly, starting with a strictly unital quasi-isomorphism

$$g_r : A^{\otimes(r-1)} \otimes N' \rightarrow N \quad (2.5.11)$$

construct a chain map

$$\tilde{g} : (M \otimes_A N')_n \rightarrow (M \otimes_A N)_n \quad (2.5.12)$$

Filter each of the complexes by length of tensor product, for instance:

$$(M \otimes_A N)_n \supset (M \otimes_A N)_{n-1} \supset \dots \supset (M \otimes_A N)_1 = M \otimes N \supset 0. \quad (2.5.13)$$

\tilde{f} respects the filtrations; it therefore induces a sequence of maps on the associated graded, that, by construction, are quasi-isomorphisms; thus, for instance by an inductive application of the 5-lemma, \tilde{f} is a quasi-isomorphism. Similarly, \tilde{g} is a quasi-isomorphism. \square

2.5.3 Rank Inequalities

We are now ready to prove the weak analogue of the inequalities used by Ishida.

Proposition 2.5.4. *For any two Lagrangians L_0 and L_1 , and any Lagrangian sphere L , and all integers $n \neq 0$*

$$hf(\tau_L^n(L_0), L_1) + hf(L_0, L_1) \geq hf(L, L_1) \cdot hf(L_0, L). \quad (2.5.14)$$

Further, if $L \not\cong L_0, L_1$ in the Fukaya category, and $|n| \geq 2$,

$$hf(\tau_L^n(L_0), L_1) + hf(L_0, L_1) \geq 2hf(L, L_1) \cdot hf(L_0, L). \quad (2.5.15)$$

Proof. First note that

$$hf(\tau_L^n L_0, L_1) = hf(L_0, \tau_L^{-n} L_1) = hf(\tau_L^{-n} L_1, L_0) \quad (2.5.16)$$

so it is enough to prove the claims for $n > 0$. From our description of $T_L^n L_0$ as a cone (section 2.5.1), we know we have an exact sequence

$$\dots \rightarrow Hom_{T_w \mathcal{A}}(C_L^n L_0, L_1) \rightarrow Hom_{T_w \mathcal{A}}(L_0, L_1) \rightarrow Hom_{T_w \mathcal{A}}(\tau_L^n L_0, L_1) \rightarrow \dots \quad (2.5.17)$$

Taking ranks gives

$$hf(\tau_L^n L_0, L_1) + hf(L_0, L_1) \geq rk(Hom_{T_w \mathcal{A}}(C_L^n L_0, L_1)) \quad (2.5.18)$$

so it is enough to show that

$$rk(Hom_{T_w \mathcal{A}}(C_L^n L_0, L_1)) \geq hf(L, L_1) \cdot hf(L_0, L_1). \quad (2.5.19)$$

By Lemma 2.5.2,

$$rk(Hom_{T_w \mathcal{A}}(C_L^n L_0, L_1)) = rk(H(hom(L, L_0) \otimes_A hom(L_1, L)))_n. \quad (2.5.20)$$

By Lemma 2.5.3, we are free to use any representatives of the quasi-isomorphism classes of $hom(L, L_0)$ and $hom(L_1, L)$. Replace them by modules of the form described by Proposition 2.3.3, say M and N . For simplicity, assume M and N are minimal. The chain complex $(M \otimes_A N)_n$ decomposes as a direct sum of complexes of the form

- $(\mathbb{Z}_2 \otimes_A \mathbb{Z}_2)_n$ (a)
- $(R_k \otimes_A \mathbb{Z}_2)_n$ (b)
- $(\mathbb{Z}_2 \otimes_A \mathcal{L}_k)_n$ (b')
- $(R_j \otimes_A \mathcal{L}_k)_n$ (c)

obtained from all the possible pairings of a summand of M and a summand of N . We want to show that

$$rk H((M \otimes_A N)_n) \geq rk M \times rk N \quad (2.5.21)$$

It is enough to prove this for each of the direct summands itemized above.

(a) The differential on $(\mathbb{Z}_2 \otimes_A \mathbb{Z}_2)_n$ is zero, so its cohomology has rank n .

(b & b') The cases $(R_k \otimes_A \mathbb{Z}_2)_n$ and $(\mathbb{Z}_2 \otimes_A \mathcal{L}_k)_n$ are clearly symmetric; without loss of generality, let us consider the first one. Let u be the generator of \mathbb{Z}_2 .

The elements $r_k^0 \otimes u$ and $r_k^1 \otimes \underbrace{\epsilon \otimes \dots \otimes \epsilon}_{n-1} \otimes u$ survive passing to cohomology and give distinct classes. Suppose additionally that $n \geq 2$ and $L \not\cong L_0, L_1$ in the Fukaya category. By Corollary 2.2.9, we must have $k \geq 3$. In this case, the generators $r_k^0 \otimes \epsilon \otimes u$ and $r_k^1 \otimes \underbrace{\epsilon \otimes \dots \otimes \epsilon}_{n-2} \otimes u$ also survive; the four classes are linearly independent.

(c) We want to show that the cohomology of $(R_j \otimes_A \mathcal{L}_k)_n$ has rank at least 4. Without loss of generality, $j \leq k$.

$r_j^0 \otimes l_k^0$ and $r_j^1 \otimes \underbrace{\epsilon \otimes \dots \otimes \epsilon}_{n-1} \otimes l_k^1$ give two classes.

- If $n \geq k$,

$$d(r_j^1 \otimes \underbrace{\epsilon \dots \epsilon}_{n-1} \otimes l_k^0) = d(r_j^0 \otimes \underbrace{\epsilon \dots \epsilon}_{n-1-k+j} \otimes l_k^1) = r_j^1 \otimes \underbrace{\epsilon \dots \epsilon}_{n-k} \otimes l_k^1.$$

Thus $r_j^1 \otimes \epsilon \dots \epsilon \otimes l_k^0 + r_j^0 \otimes \epsilon \dots \epsilon \otimes l_k^1$, the sum of the elements above, is in the kernel; it gives a non-zero cohomology class. Also,

$$d(r_j^0 \otimes \underbrace{\epsilon \dots \epsilon}_{k-1} \otimes l_k^0) = r_j^0 \otimes l_k^1 + r_j^1 \otimes \underbrace{\epsilon \dots \epsilon}_{k-j} \otimes l_k^0 \quad (2.5.22)$$

both summands of which lie in the kernel of d . Under d , they are only the image of $r_j^0 \otimes \epsilon \dots \epsilon \otimes l_k^0$. This gives us the final class we wanted for the cohomology. See figure 2-3 for the case $j = 2, k = 3$ and $n = 3$.

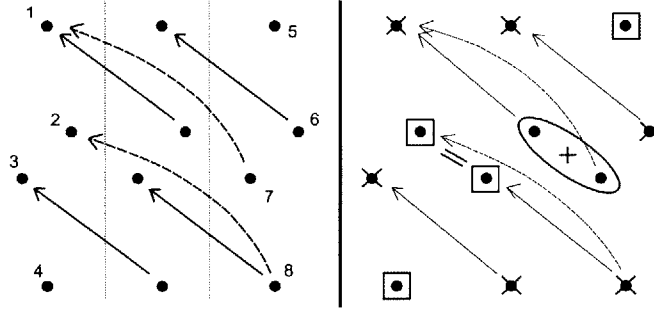


Figure 2-3: $(R_2 \otimes_A \mathcal{L}_3)_3$: generators for the complex, differentials, and generators for the cohomology. In both diagrams, each dot is a generator for $(R_2 \otimes_A \mathcal{L}_3)_3$; the i^{th} column contains the generators corresponding to $R_3 \otimes \epsilon \otimes \dots \otimes \epsilon \otimes \mathcal{L}_3$, with $i - 1$ copies of ϵ . For instance, suppressing the subscripts 2 and 3: the dot labelled 1 is $r^1 \otimes l^1$, 2 is $r^0 \otimes l^1$, 3 is $r^1 \otimes l^0$, 4 is $r^0 \otimes l^0$; the dot labelled 5 corresponds to $r^1 \otimes \epsilon \otimes \epsilon \otimes l^1$, and similarly for 6, 7 and 8. The arrows give the differential; the dashed ones come from $\mu^3(\epsilon, \epsilon, \cdot)$ on \mathcal{L}_3 , and the full ones from $\mu^2(\cdot, \epsilon)$ on R_2 . The right-hand side diagram gives generators for the cohomology, which has dimension exactly four.

- If $j \leq n < k$, the classes $r_j^0 \otimes l_k^1$ and $r_j^1 \otimes \underbrace{\epsilon \dots \epsilon}_{n-1} \otimes l_k^0$ survive to cohomology.
- If $n < j$, the differential on $(R_j \otimes_A \mathcal{L}_k)_n$ is trivial, and the (co)homology has rank $4n$.

In all cases, two of the classes are at the ‘start’ of the complex ($r_j^0 \otimes l_k^0$ and $r_j^0 \otimes l_k^1$ or its equivalence class), and two at the ‘end’ (e.g. $r_j^1 \otimes \underbrace{\epsilon \otimes \dots \otimes \epsilon}_{n-1} \otimes l_k^1$ and $r_j^1 \otimes \underbrace{\epsilon \dots \epsilon}_{n-1} \otimes l_k^0$, or its equivalence class). Suppose additionally that $n \geq 2$ and, and neither L_0 nor L_1 is quasi-isomorphic to L in the Fukaya category. By Corollary 2.2.9, $k \geq j \geq 3$; it is then easy to check

that there are actually at least 8 classes that survive. The strategy is similar to that in (b) and (b'), and we shall not go through each case in detail; the point is that you can add two 'front'-like classes, shifted by an inserted ϵ , and two 'back'-like classes, shifted by deleting one of the $(n - 1)$ ϵ 's that appear in the expressions for them. See figures reffig:j3k3n4 through 2-6 for examples: the cases where (j, k, n) is equal to $(3, 3, 4)$, $(3, 4, 4)$ and $(3, 4, 5)$.

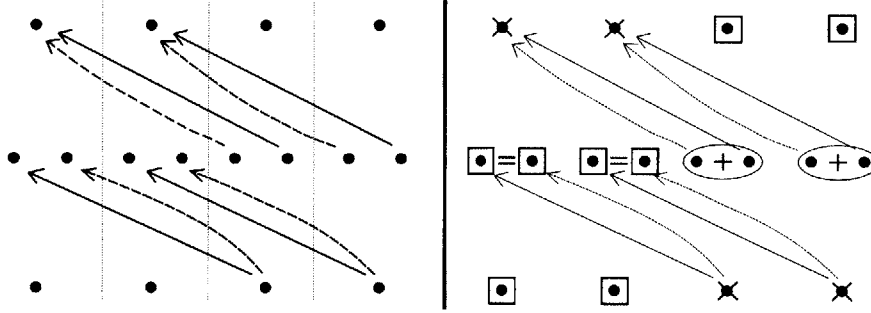


Figure 2-4: $(R_3 \otimes_A \mathcal{L}_3)_4$: generators for the complex, differentials, and generators for the cohomology.

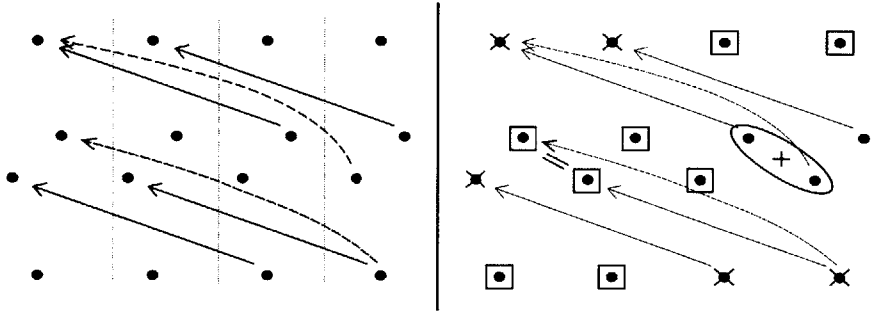


Figure 2-5: $(R_3 \otimes_A \mathcal{L}_4)_4$: differentials, and generators for the cohomology.

□

Remark 2.5.5. *Suppose that we started with weaker hypothesis, as in Remark 2.4.6; one could proceed analogously to above to get the inequality*

$$hf(\tau_L^n, L_0, L_1) + hf(L_0, L_1) \geq rkH((\text{hom}(L, L_0) \otimes_B \text{hom}(L_1, L))_n). \quad (2.5.23)$$

However, we cannot in general obtain similar lower bounds for the right-hand side.

2.6 Conclusion of argument

From here on, the proof of Theorems 1.1.1 and 1.1.2 closely follows Ishida's argument. Rank of Lagrangian Floer cohomology plays the role of intersection numbers, except we have to be careful whenever it is 2. To get the conclusion of Theorem 1.1.1, we will be repeatedly using

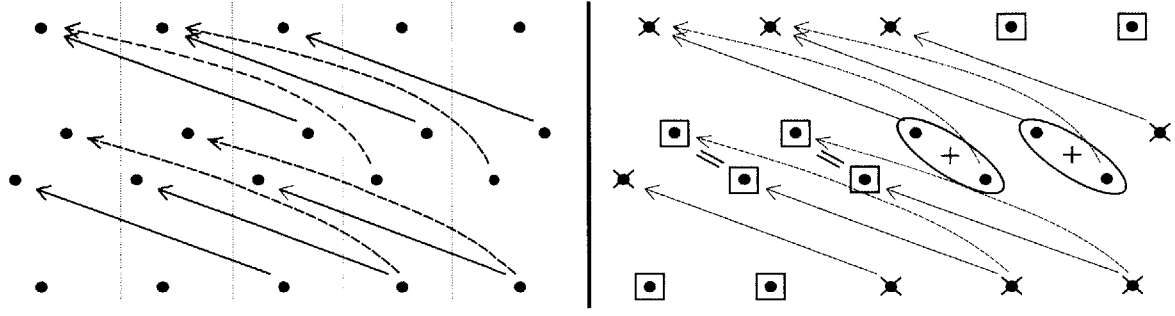


Figure 2-6: $(R_3 \otimes_A \mathcal{L}_4)_5$: generators for the complex, differentials, and generators for the cohomology.

the fact that if a symplectomorphism ϕ of M is symplectically isotopic to the identity Id , then for any Lagrangian sphere L , $\phi(L)$ is exact Lagrangian isotopic to L .

Lemma 2.6.1. (see [35, Lemma 2.3]) *Let L , L_0 and L_1 be Lagrangians such that L is a sphere, $L \not\cong L_0$ in the Fukaya category, and $hf(L, L_0) \geq 2$. Then for all $n \neq 0$,*

$$hf(L, L_1) > hf(L_0, L_1) \Rightarrow hf(L, \tau_L^n(L_1)) < hf(L_0, \tau_L^n(L_1)) \quad (2.6.1)$$

Proof. Replacing n by $-n$, the statement is equivalent to

$$hf(L, L_1) > hf(L_0, L_1) \Rightarrow hf(L, L_1) < hf(\tau_L^n(L_0), L_1) \quad (2.6.2)$$

As $hf(L_0, L) \geq 2$, from Proposition 2.5.4 we have

$$2hf(L, L_1) - hf(L_0, L_1) \leq hf(\tau_L^n(L_0), L_1) \quad (2.6.3)$$

which, assuming $hf(L, L_1) - hf(L_0, L_1) > 0$, gives

$$hf(L, L_1) < hf(\tau_L^n(L_0), L_1) \quad (2.6.4)$$

as required. \square

We are now ready to conclude the proofs.

Case $hf(L, L') > 2$.

As $hf(L, L') > hf(L', L')$, Lemma 2.6.1 immediately implies that we cannot have $\tau_L^n = 1$ for any $n \neq 0$. The same holds for $\tau_{L'}$. Thus, if the group generated by τ_L and $\tau_{L'}$ is not free, there must exist $k \in \mathbb{N}$ and $a_i, b_i \in \mathbb{Z}^*$, $1 \leq i \leq n$, such that

$$\tau_{L'}^{b_n} \tau_L^{a_n} \dots \tau_{L'}^{b_1} \tau_L^{a_1} = 1. \quad (2.6.5)$$

Notice that by assumption, $hf(L', \tau_L^{a_1} L) > hf(L, \tau_L^{a_1} L) = 2$. By Lemma 2.6.1,

$$hf(L', \tau_{L'}^{b_1} \tau_L^{a_1} L) < hf(L, \tau_{L'}^{b_1} \tau_L^{a_1} L). \quad (2.6.6)$$

We may then use Lemma 2.6.1 to see that

$$hf(L', \tau_L^{a_2} \tau_{L'}^{b_1} \tau_L^{a_1} L) > hf(L, \tau_L^{a_2} \tau_{L'}^{b_1} \tau_L^{a_1} L). \quad (2.6.7)$$

Repeated iterations give

$$hf(L', \tau_{L'}^{b_k} \dots \tau_L^{a_1} L) < hf(L, \tau_{L'}^{b_k} \dots \tau_L^{a_1} L) \quad (2.6.8)$$

where we recognise the left-hand side to be $hf(L', L)$ and the right-hand side to be $hf(L, L)$, a contradiction.

Case $hf(L, L') = 2$.

The key will be:

Claim 2.6.2. *For all $m \neq 0$, we have $hf(L', \tau_{L'}^m L) < hf(L, \tau_{L'}^m L)$.*

Notice that this immediately implies that for all $m \neq 0$, $\tau_{L'}^m \neq 1$. Similarly, $\tau_L^m \neq 1$ for all such m . Moreover, the statement is equivalent to

$$hf(L', \tau_{L'}^{b_1} \tau_L^{a_1} L) < hf(L, \tau_{L'}^{b_1} \tau_L^{a_1} L) \quad (2.6.9)$$

for all $a_1 \in \mathbb{Z}$, $b_1 \in \mathbb{Z}^*$, which is the second inequality we got in the iteration we conducted for the first case. We may then proceed in exactly the same way, repeatedly using Lemma 2.6.1 to get that

$$hf(L', \tau_{L'}^{b_k} \dots \tau_L^{a_1} L) < hf(L, \tau_{L'}^{b_k} \dots \tau_L^{a_1} L) \quad (2.6.10)$$

which is a contradiction.

Thus, to prove that the subgroup generated by τ_L and $\tau_{L'}$ is free, it remains only to prove the claim.

Proof. of claim 2.6.2. We treat the cases $m = 1$ and $m > 1$ separately. For $m > 1$, the second part of Lemma 2.6.1 gives

$$hf(\tau_{L'}^m L, L) \geq 2hf(L', L) \cdot hf(L, L') - hf(L, L) \quad (2.6.11)$$

where the right-hand side is equal to 6, so we are done.

For $m = 1$, we are certainly done unless $hf(\tau_{L'} L, L) = 2$. If this is the case, use the exact sequence:

$$\dots \rightarrow HF(L, L) \rightarrow HF(L, \tau_{L'}^{-1} L) \rightarrow HF(L', L) \otimes HF(L, L') \rightarrow \dots \quad (2.6.12)$$

Considering ranks, this must split as a short exact sequence

$$0 \rightarrow HF(L, \tau_{L'}^{-1} L) \rightarrow HF(L, L') \otimes HF(L', L) \rightarrow HF(L, L) \rightarrow 0. \quad (2.6.13)$$

In particular, multiplication $HF(L, L') \otimes HF(L', L) \rightarrow HF(L, L)$ is surjective. By Corollary 2.2.6, L and L' are quasi-isomorphic in the Fukaya category, a contradiction. \square

2.7 Examples: some Milnor fibres

We first explain what we mean by a Milnor fibre of a hyperplane singularity, as a symplectic manifold, and establish some relationships between fibres of a fixed singularity (section 2.7.1). Section 3.1.1 discusses adjacency of singularities. We then collect material on Khovanov and Seidel's framework for studying fibres of (A_m) singularities, and certain Lagrangian spheres in them (section 2.7.3). Subsection 2.7.4 presents the actual examples, constructed using this framework and adjacency arguments.

2.7.1 Milnor fibres as symplectic manifolds

Let f be a non-constant polynomial $\mathbb{C}^{n+1} \rightarrow \mathbb{C}$; suppose that $f(0) = 0$, and that f has an isolated critical point at 0. Assume that $n \geq 2$.

Theorem. (Milnor [51, Theorems 4.8 and 5.11]) *Fix f as above. There exists ϵ_f such that for all $\epsilon < \epsilon_f$, the sphere S_ϵ intersects $f^{-1}(0)$ transversely, and the space $S_\epsilon \setminus f^{-1}(0)$ is a smooth fibre bundle over S^1 , with projection mapping*

$$\phi(z) = f(z)/|f(z)|.$$

The class of the fibre, as a smooth manifold, is independent of ϵ . Moreover, given ϵ , there exists $\delta(\epsilon)$ such that for all c with $0 \neq |c| < \delta(\epsilon)$, the complex hypersurface $f^{-1}(c)$ intersects S_ϵ transversely, and $f^{-1}(c) \cap B_\epsilon$, where B_ϵ is the open ϵ -ball, is a smooth manifold which is diffeomorphic to the fibre $\phi^{-1}(\arg(c))$.

Milnor [51] also shows that $f^{-1}(c) \cap B_\epsilon$ is homotopy equivalent to a finite bouquet of half-dimensional spheres. The count of these is called the Milnor number of the singularity; there is an algebraic expression for it in terms of the partial derivatives of f .

We may assume that $\delta(\epsilon)$ increases monotonely for $\epsilon \in (0, \epsilon_f)$. Let $\delta(\epsilon_f)$ be their supremum. We say $c \in \mathbb{C}$ is admissible if $|c| < \delta(\epsilon_f)$. For a given c , we define its admissible range to be the interval consisting of all ϵ such that $c < \delta(\epsilon)$. Note that as c tends to zero, the lower end-point of the admissible range for c does too.

Assume c is admissible, and that ϵ is in the admissible range of c . Define

$$F_{c,\epsilon} := f^{-1}(c) \cap \overline{B}_\epsilon \tag{2.7.1}$$

where \overline{B}_ϵ be the closed ball of radius ϵ . When it is clear which ϵ we are using, we will sometimes denote this simply by F_c ; if there are several polynomials involved, we will use the notation F_c^f or $F_{c,\epsilon}^f$. For any admissible c , there exists $\epsilon' > \epsilon$ that is also admissible; if $p \in \partial F_{c,\epsilon}$, our convention is that $T_p F_{c,\epsilon}$ means $T_p F_{c,\epsilon'}$. Let $\theta = i/4 \sum_{i=0}^n (z_i d\bar{z}_i - \bar{z}_i dz_i)$ and $\Omega = d\theta$ be the usual forms on \mathbb{C}^{n+1} . Let Z be the associated negative Liouville vector field: $i_Z \Omega = -\theta$, and ϕ_t its flow, where t parametrizes time.

The restrictions of θ and Ω , say θ_c and ω_c , give each $F_{c,\epsilon}$ the structure of an exact symplectic manifold. We shall call any such manifold the 'Milnor fibre of f '. Let Z_c be the negative Liouville vector field on $F_{c,\epsilon}$. Notice $-Z_c$ is the gradient of $\sum_{i=0}^n |z_i|^2$ with respect to the usual Kähler metric; this points outwards along level-sets. Thus, as $f^{-1}(c) \cap S_\epsilon$, Z_c points inwards along the boundary of $F_{c,\epsilon}$. We denote the negative Liouville flow on each fibre by ϕ_t^{fib} .

Following a well-established approach for exact symplectic manifolds with contact type boundary, we glue cylindrical ends to $F_{c,\epsilon}$ to obtain a non-compact symplectic manifold $\overline{F}_{c,\epsilon}$:

$$(\overline{F}_{c,\epsilon}, \overline{\omega}_c, \overline{\theta}_c) = (F_{c,\epsilon}, \omega_c, \theta_c) \cup (\partial F_{c,\epsilon} \times \mathbb{R}^+, d(e^t \theta_c|_{\partial F_{c,\epsilon}}), e^t \theta_c|_{\partial F_{c,\epsilon}}) \quad (2.7.2)$$

where t is the coordinate on \mathbb{R}^+ , and the gluing is made using the negative Liouville flow ϕ_t^{fib} . For a fixed c , this construction is independent of the choice of admissible ϵ ; hereafter we will often suppress the subscript ϵ .

Lemma 2.7.1. *Suppose c_1 and c_2 are admissible for f . Then there is an exact symplectomorphism between \overline{F}_{c_1} and \overline{F}_{c_2} . More precisely, suppose that ϵ is any constant in $(0, \epsilon_f)$ that lies in the admissible range for both c_1 and c_2 (notice that we can always find such an ϵ). To any smooth path in $B_{\delta(\epsilon)} \setminus 0$, with end-points at c_1 and c_2 , we associate a (non-canonical) exact symplectomorphism of the completed fibres, defined up to Hamiltonian isotopy.*

Fix $\epsilon < \epsilon_f$, and let

$$E = \bigcup_{|c| < \delta(\epsilon)} F_{c,\epsilon}. \quad (2.7.3)$$

Let $\partial E \subset E$ be its ‘horizontal’ boundary, i.e. the union of the boundaries of all the $F_{c,\epsilon}$. (When talking about the tangent space to E at a point of ∂E , we use the same convention as for $F_{c,\epsilon}$.) Putting together the negative Liouville flows on each fibre gives smooth maps $E \rightarrow E$, that we shall also denote ϕ_t^{fib} ; the associated vector field (Z^c on each fibre) will be called Z^{fib} .

Fix c with $|c| < \delta(\epsilon)$. For every $p \in F_c$, the vector space $T_p F_c \subset T_p E$ has a canonical complement, given by taking the symplectic orthogonal to $T_p F_c$. This gives a ‘horizontal tangent space’ for our fibration: every vector in $T_{f(p)} B_{\delta(\epsilon)}$ has a preferred lift in $T_p E$. Fix a smooth path $\gamma : [0, 1] \rightarrow B_{\delta(\epsilon)} \setminus 0$. The ‘symplectic parallel transport’ associated to the horizontal tangent space is a priori only defined on a (possibly empty) subset of $F_{\gamma(0)}$: at points of the boundary ∂E , the horizontal tangent space may not lie in the tangent space of ∂E . By construction, we have the following:

Lemma 2.7.2. *Symplectic parallel transport defines an exact symplectomorphism from its domain in $F_{\gamma(0)}$ to its image in $F_{\gamma(1)}$.*

Note that we could carry out this process for any symplectic form on E that restricts to one on the fibres, and that it is enough for the form to be closed and non-degenerate on each fibre.

Considering volumes, one cannot hope in general for symplectomorphisms defined on the whole of each fibre. Instead, we shall work with the completed fibres. The key is:

Lemma 2.7.3. *We can construct a closed form Ω'' on E such that*

1. *for every c , $\Omega''|_{F_c} = \Omega|_{F_c}$;*
2. *the horizontal tangent space determined by Ω'' is invariant under ϕ_t^{fib} in some collar neighbourhood of the boundary ∂E ;*
3. *Ω'' and Ω agree outside a collar neighbourhood of ∂E .*

To prove Lemma 2.7.1, let us replace Ω by such an Ω'' , and work instead with the horizontal tangent space given by Ω'' . Property 2 then allows us to extend the corresponding parallel transport to the completed fibres, in the obvious fashion.

Proof. of Lemma 2.7.3. There exists $\tau > 0$ such that Z^{fib} is non zero at all points of

$$U = \bigcup_{t \in [0, \tau]} \phi_t^{fib}(\partial E) \subset E. \quad (2.7.4)$$

We will use the identification

$$\begin{aligned} \partial E \times [-\tau, 0] &\cong U \\ (a, t) &\mapsto \phi_{-t}^{fib}(a). \end{aligned} \quad (2.7.5)$$

Let $\alpha = \theta|_{\partial E}$. Define $\theta' \in \Omega^1(U)$ by

$$\theta' = e^t(\phi_t^{fib})^* \alpha \quad (2.7.6)$$

where $t \in [-\tau, 0]$. By construction, $\theta'|_{F_c} = \theta|_{F_c}$, for all fibres F_c . The form $\Omega' = d\theta' \in \Omega^2(U)$ satisfies conditions one and two.

Let $\xi = \theta' - \theta \in \Omega^1(U)$. Let ψ be a smooth cut-off function on $[-\tau, 0]$ such that $\psi = 1$ on $[-\tau/2, 0]$ and $\psi = 0$ on $[-\tau, -2\tau/3]$. This induces a function $U \rightarrow \mathbb{R}$, that we also denote by ψ . Set

$$\Omega'' = \Omega + d(\psi\xi) \in \Omega^2(U). \quad (2.7.7)$$

This agrees with Ω' on an collar neighbourhood on ∂E , and with Ω outside a (larger) collar neighbourhood of ∂E , and so satisfies conditions 2 and 3. As ξ vanishes on fibres, condition one is also satisfied. \square

Holomorphic reparametrization. So far, we have treated f simply as the germ at the origin of a function $\mathbb{C}^{n+1} \rightarrow \mathbb{C}$; from the point of view of singularity theory, it is more natural to think of f as a representative of its equivalence class under biholomorphic change of coordinates (preserving the origin). What can we say for the corresponding (completed) fibres?

Lemma 2.7.4. *Say $f = g \circ h$, some holomorphic change of coordinates h . Then there is an exact symplectic embedding from a Milnor fibre of f to a completed Milnor fibre of g , and vice-versa.*

Proof. First note that h maps (subsets of) Milnor fibres of f to (subsets of) Milnor fibres of g . There exists ϵ'_f such that $h(B_{\epsilon'_f}) \subset B_{\epsilon_g}$. Fix c such that $|c| < \delta(\epsilon'_f)$. h maps F_c^f into F_c^g , and thus gives an embedding $F_{c,f} \hookrightarrow \overline{F}_{c,g}$; this is not in general symplectic. However, notice that both symplectic forms are compatible with J . By connecting the associated metrics, we can find a path connecting the symplectic forms. Also, note that any closed two-form is automatically exact. Thus we can use a Moser argument to deform our embedding to a symplectic one. By homology considerations, this symplectic embedding is then automatically exact (notice that we are using $n \geq 2$). \square

Remark 2.7.5. *A more careful set-up gives exact symplectomorphisms between completed Milnor fibres of f and g (see Lemma 3.1.5). The above will suffice for Corollary 2.7.7.*

Property S. We say that a exact symplectic manifold with contact type boundary has ‘property S’ if it satisfies the hypothesis of Theorem 1.1.1: its interior contains two Lagrangian spheres L_0, L_1 such that $hf(L_0, L_1) \geq 2$ and L_0 and L_1 are not Fukaya isomorphic.

Lemma 2.7.6. *Suppose there’s an exact symplectic embedding from a Milnor fibre F to a completed Milnor fibre $\overline{F'}$ (these needn’t be fibres of the same polynomial), and, moreover, that F has property S. Then F' has property S.*

Proof. Let $\iota : F \hookrightarrow \overline{F'}$ be the exact symplectic embedding, and $L_0, L_1 \subset F$ the Lagrangian spheres given by property S. Let $\overline{F'_T} = F' \cup \partial F' \times [0, T]$ be a truncation of the completed Milnor fibre; this is an exact symplectic manifold with contact type boundary. Choose T large enough such that $\iota(F) \subset \text{int}(\overline{F'_T})$. It follows from Lemma 2.2.4 that $hf(\iota(L_0), \iota(L_1)) = hf(L, L')$, and $\iota(L_0)$ and $\iota(L_1)$ are not isomorphic in $\mathcal{Fuk}(\overline{F'_T})$. Flow $\iota(L_0)$ and $\iota(L_1)$ by the negative Liouville vector field inside $\overline{F'_T}$ until their images, say L'_0 and L'_1 , lie in the interior of F' . L'_0 is quasi-isomorphic to L_0 in $\mathcal{Fuk}(\overline{F'_T})$, and similarly for L'_1 and L_1 . Thus L'_0 and L'_1 are not quasi-isomorphic, and $hf(L'_0, L'_1) = hf(L_0, L_1)$. Using Lemma 2.2.4 again, we see that this is also true in the Fukaya category of F' . \square

The following is now immediate.

Corollary 2.7.7. *If F_c , any Milnor fibre of a polynomial f , has property S, then so do all completed Milnor fibres of f . Moreover, if g is another representative of the equivalence class of the germ of f , then all Milnor fibres of g have property S.*

2.7.2 Adjacency of singularities and symplectic embeddings

Hereafter an ‘isolated singularity’ $[f]$ will be the germ of a polynomial $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$, with $f(0) = 0$ and an isolated singularity at 0, up to biholomorphic change of coordinates (fixing the origin).

Definition 2.7.8. *Let $[f]$ and $[g]$ be isolated singularities. $[f]$ is said to be adjacent to $[g]$ if there exist arbitrarily small polynomial perturbations p_k , $k \in \mathbb{N}$, such that $[f + p_k] = [g]$.*

Adjacency of singularities is a well-studied classical topic; we shall use material collected in the survey [7].

Lemma 2.7.9. *Suppose that $[f]$ and $[g]$ are isolated singularities such that $[f]$ is adjacent to $[g]$, and that $n \geq 2$. Then there exists an exact symplectic embedding from a Milnor fibre of f to a completed Milnor fibre of g .*

Proof. Consider the map

$$\begin{aligned} q : \mathbb{C}^{n+1} \times \mathbb{C} &\rightarrow \mathbb{C} \times \mathbb{C} \\ q(\mathbf{z}, \gamma) &= (f(\mathbf{z}) + \gamma p_k(\mathbf{z}), \gamma) \end{aligned} \tag{2.7.8}$$

There exists an $n \in \mathbb{N}$ and $\epsilon < \epsilon_f$ such that for all $(\mathbf{z}, \gamma) \in B_{\delta(\epsilon)} \times B_2$, $q^{-1}(c, \gamma) \cap S_\epsilon$, where S_ϵ is the sphere in $\mathbb{C}^{n+1} \times \gamma$. Let $Q_{(c, \gamma)} = q^{-1}(c, \gamma) \cap \overline{B}_\epsilon$. Let H be the set of points (c, γ)

in $B_{\delta(\epsilon)} \times B_2$ such that $Q_{(c,\gamma)}$ is singular. H is an algebraic curve; in particular, this means that $(B_{\delta(\epsilon)} \times B_2) \setminus H$ is path-connected. The $Q_{(c,\gamma)}$ inherit an exact symplectic structure from \mathbb{C}^{n+2} , and can be extended to non-compact symplectic manifolds $\overline{Q}_{(c,\gamma)}$ by gluing conical ends, in a similar fashion to the \overline{F}_c . For sufficiently small c , $Q_{(c,1)}$ contains a Milnor fibre of a representative of $[g]$ as a subset: if $[f] \neq [g]$, we certainly have $\epsilon_{f+p_k} \leq e$. $Q_{(c',0)}$ is a Milnor fibre of f . Pick a smooth path in $(B_{\delta(\epsilon)} \times B_2) \setminus H$ connecting $(c, 1)$ and $(c', 0)$. Adapting the construction used to prove Lemma 2.7.1, we get exact symplectomorphisms from $\overline{Q}_{(c,1)}$ to $\overline{Q}_{(c',0)}$, completing the proof. □

Corollary 2.7.10. *Suppose that $[f]$ is adjacent to $[g]$, and that $[g]$ has property S . Then $[f]$ also has property S .*

Proof. This follows immediately from Lemmata 2.7.6 and 2.7.9. □

2.7.3 Lagrangian spheres in (A_m) fibres

The singularity of type (A_m) is the one associated to the polynomial $z_0^2 + \dots + z_n^2 + z_{n+1}^{m+1}$; we are interested in the case $n \geq 2$. For any m , [42] describes how to associate Lagrangian submanifolds of the (A_m) fibre to certain curves on the unit disc with $(m+1)$ marked points, which we denote by D_{m+1} . If the curve intersects the marked points exactly twice, once at both endpoints, one gets a Lagrangian sphere, defined up to Lagrangian isotopy. Also, an isotopy of the curve, relative to the marked points, gives a Lagrangian isotopy of the sphere.

Suppose the marked points are aligned horizontally. A basis for the homology of the fibre is given by the spheres corresponding to the straight-line segments between consecutive marked points. (See figure 2-7 for the case $m = 2$.)

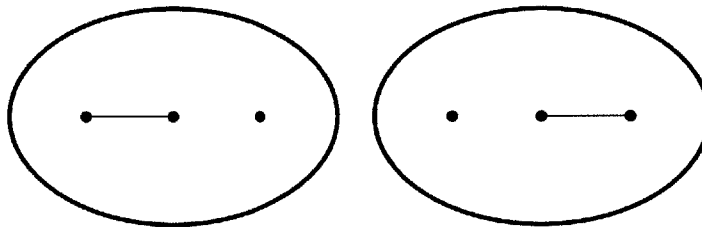


Figure 2-7: Curves corresponding to a basis for homology for the (A_2) fibre, say a and b .

A lot of information about these spheres can be read straight from the curves in D_{m+1} . Notably, the rank of the Floer cohomology between any two such spheres is twice the geometric intersection number of the curves they are associated to [42, Theorem 1.3]. For two curves that intersect ‘minimally’ (see [42]), this number counts $1/2$ for a common end-point, and 1 for all other intersection points.

It is also easy to determine the homology class of a sphere. For even n , it only depends on the end-points of the corresponding curve, and orientation; any two curves with common end-points actually give smoothly isotopic spheres [49, Proposition 5.1]. Given a sphere associated

to a curve in D_{m+1} , the action of the Dehn twist about that sphere can be described in terms of a half-twist of D_{m+1} , preserving the marked points [50, Lemma 7.1]. To compute homology classes of spheres when n is odd, one needs to use this together with the Picard-Lefschetz theorem ([53, 46]; see e.g. [7, section 2.1] for a concise account). This applies to all the Lagrangian spheres considered here; for a fixed sphere S , it describes the action of the Dehn twist τ_S on a class x in the middle homology of the Milnor fibre:

$$\tau_S(x) = x + (-1)^{n(n+1)/2}(x \circ S)S. \quad (2.7.9)$$

2.7.4 Some families of examples

Even complex dimension

There are many examples of pairs of Lagrangian spheres in the (A_2) fibre realising property S that can be constructed using the set-up of [42]. Moreover, one can arrange for them to be smoothly isotopic by using the fact that it is enough for them to share end-points.

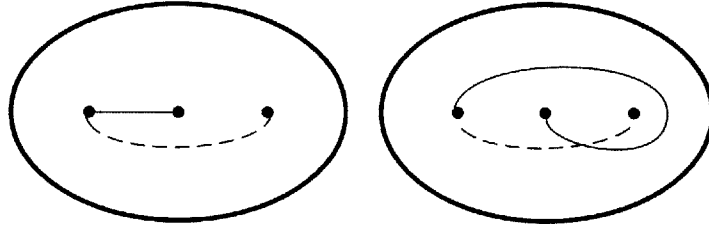


Figure 2-8: Even case: curves corresponding to two homologous spheres in the (A_2) fibre.

Arguably the simplest example is given in figure 2-8; the two full-stroke curves correspond to two isotopic spheres, say L_0 and L_1 . The corresponding curves only intersect at their end-points, so by [42, Theorem 1.3], $hf(L_0, L_1) = 2$. Consider the sphere associated to the dashed curves, say L_2 . We have $hf(L_0, L_2) = 1$ and $hf(L_1, L_2) = 3$; thus L_0 and L_1 are not Fukaya isomorphic. They satisfy the hypothesis of Theorem 1.1.1: (A_2) has property S .

Lemma 2.7.11. *Any degenerate singularity is adjacent to (A_2) .*

Proof. By the parametric Morse lemma (see e.g. [7, Section 1.3]), a degenerate singularity is equivalent to a function of the form

$$\phi(z_0, \dots, z_k) + z_{k+1}^2 + \dots + z_n^2 \quad (2.7.10)$$

where $\phi \in \mathcal{M}^3$, and $\mathcal{M} \subset \mathbb{C}[z_0, z_1, \dots, z_n]$ is the maximal ideal. For sufficiently small $\epsilon > 0$,

$$\phi(z_0, \dots, z_k) + \epsilon(z_1^2 + z_2^2 + \dots + z_k^2) + z_{k+1}^2 + \dots + z_n^2 \quad (2.7.11)$$

has an isolated singularity at zero. Moreover, by construction, it has corank one. The parametric Morse lemma implies that it is an (A_m) singularity, for some $m \geq 3$; in particular, it is adjacent to (A_3) [7, Section 2.7]. \square

Thus, by Corollary 2.7.10, the Milnor fibre F of a degenerate singularity has property S ; moreover, the two Lagrangian spheres realising this can be chosen to be smoothly isotopic, as they come from smoothly isotopic spheres in (A_2) . This implies that the Dehn twists about them agree as elements of $\pi_0(\text{Diff}^+ F)$.

Odd complex dimension

One can still find examples of spheres in the (A_2) fibre that satisfy the hypothesis of Theorem 1.1.1, e.g. those above. (The calculation of the rank of Floer cohomology is independent of dimension.) However, we cannot find examples where they are also homologous.

Lemma 2.7.12. *If two curves in D_3 correspond to homologous spheres in the (A_2) fibre, then they are isotopic relative to the marked points.*

Proof. Start, exceptionally, by considering the case where the fibre is one-dimensional. Let C_0 and C_1 be the two curves in D_3 , and s_0 and s_1 be the corresponding circles. For this case, one could carry out a ‘by hand’ proof, starting by analysing the contributions of various portions of the curve to the homology classes corresponding to a and b (as introduced in Figure 2-7).

Alternatively, let us make use of some classical results (see e.g. [14, 13]). The (A_2) fibre, a once-punctured torus, is a double cover of D_3 , branched over the marked points. Denote it by A_2^1 . Let $\text{Diff}^+ D^3$ be the group of orientation-preserving diffeomorphisms of the disc that preserve the three marked points; $\text{Diff}_c^+ D^3$ the subgroup of such diffeomorphisms that are compactly supported; similarly with $\text{Diff}^+ A_2^1$ and $\text{Diff}_c^+ A_2^1$. Every element of $\text{Diff}^+ D_3$ lifts to two possible elements of $\text{Diff}^+ A_2^1$, differing by the deck action; every element of $\text{Diff}_c^+ D_3$ induces a unique element of $\text{Diff}_c^+ A_2^1$.

The half-twists in a and b are generators for an action of the braid group Br_3 on the base D_3 ; this lifts to an action on the total space, and the half-twists give the Dehn twists in the circles corresponding to a and b . We have

$$\begin{array}{ccccc} PSL_2(\mathbb{Z}) & \xrightarrow{\cong} & \pi_0(\text{Diff}^+ D^3) & \xleftarrow{2:1} & \pi_0(\text{Diff}^+ A_2^1) & \xrightarrow{\cong} & \pi_0(\text{Diff}^+ T^2) \cong SL_2(\mathbb{Z}) & (2.7.12) \\ & & \uparrow & & \uparrow & & & \\ Br_3 & \xrightarrow{\cong} & \pi_0(\text{Diff}_c^+ D^3) & \xrightarrow{\cong} & \pi_0(\text{Diff}_c^+ A_2^1) & & & \end{array}$$

where one can use the presentations

$$Br_3 \cong \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle \quad (2.7.13)$$

$$PSL_2(\mathbb{Z}) \cong \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2, (\sigma_1 \sigma_2)^3 = 1 \rangle \quad (2.7.14)$$

$$SL_2(\mathbb{Z}) \cong \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2, (\sigma_1 \sigma_2)^6 = 1 \rangle. \quad (2.7.15)$$

(See, for instance, the book [13]. Note that a \mathbb{Z}_2 equivariant diffeomorphism which is isotopic to the identity is \mathbb{Z}_2 equivariantly isotopic to the identity [14, Theorem 1].) In particular, the action of $SL_2(\mathbb{Z})$ on the homology of A_2^1 can be realised by \mathbb{Z}_2 equivariant maps.

The curves C_0 and C_1 can both be obtained from a (or b) through a series of forwards or backwards half-twists in a and b . Choose a map $f \in \text{Diff}^+ D^3$ mapping C_0 to C_1 . This lifts to a \mathbb{Z}_2 equivariant $\tilde{f} \in \text{Diff}^+(A_2^1)$, mapping s_0 to s_1 . Now pick a \mathbb{Z}_2 equivariant $h \in \text{Diff}^+ A_2^1$

that fixes s_1 , and such that $h \circ \tilde{f}$ acts as the identity on homology (remember that we are assuming that s_0 and s_1 are isotopic). Any element of $\pi_0(\text{Diff}^+T^2)$ is uniquely determined by its action on homology; thus $h \circ \tilde{f}$ is isotopic to the identity. By construction, it is a \mathbb{Z}_2 equivariant map; by [14, Theorem 1], it is \mathbb{Z}_2 equivariantly isotopic to the identity. This implies that C_0 and C_1 are isotopic through elements of $\text{Diff}^+(D_3)$.

Now suppose the fibre, say A_2^n , has odd dimension $n \geq 3$. Fix a curve \mathcal{C} in D_3 with boundary on the marked points, and interior disjoint from them. Let $S_{\mathcal{C}}$ be the sphere corresponding to \mathcal{C} ; say $[S_{\mathcal{C}}] = (S_a, S_b) \in H_n(A_2^n)$, with respect to the basis corresponding to a and b . One can also consider the circle $s_{\mathcal{C}} \subset A_2^1$ associated to \mathcal{C} , as discussed in the previous paragraph; say $[s_{\mathcal{C}}] = (s_a, s_b) \in H_1(A_2^1)$, also with respect to the basis corresponding to a and b . \mathcal{C} can be obtained from a (or b) through a series of forward and backward half-twists in a and b ; thus the Picard-Lefschetz theorem implies that

$$(S_a, S_b) = (\pm s_a, \pm s_b). \quad (2.7.16)$$

Reversing some orientations if needed, this allows us to conclude using the $n = 1$ case. \square

By [3, Theorem 1.3], every Lagrangian sphere in A_2^n , for $n \geq 3$, as an element of the Fukaya category, is in the braid group orbit of (either of) the Lagrangian spheres corresponding to a or b . Thus we expect Lemma 2.7.12 to imply that there can be no pair of *homologous* spheres in the (A_2) fibre satisfying the hypothesis of Theorem 1.1.1.

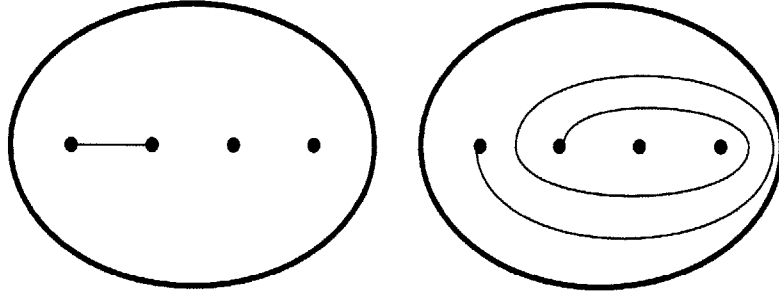


Figure 2-9: Odd case: curves corresponding to two homologous spheres in the (A_3) fibre.

One can nonetheless find pairs of homologous spheres in the (A_3) fibre satisfying the hypothesis of Theorem 1.1.1. Figure 2-9 gives an example. Let L_0 (resp. L_1) be the sphere corresponding to the left (resp. right)-hand curve. We have $hf(L_0, L_1) = 4$. Let τ_a , τ_b and τ_c be the Dehn twists in the spheres corresponding to the first, second and third straight-line segments between marked points, similarly to the (A_2) case and figure 2-7. (Note $\tau_a = \tau_{L_0}$.) One can check that

$$L_1 = \tau_b \tau_c^2 \tau_b^2 \tau_c^2 \tau_b L_0 \quad (2.7.17)$$

and, using the Picard-Lefschetz theorem, that L_0 and $\pm L_1$ are homologous (the sign depends on the *mod* 4 remainder of n). As the L_i have odd real dimension, this implies that $[L_0] \cdot [L_1] = 0$. Moreover, notice that the fibre has real dimension ≥ 6 , and is simply connected. Thus, by the Whitney trick, there is a smooth isotopy of the fibre such that the image of L_0 is disjoint from L_1 . This means that τ_{L_0} and τ_{L_1} commute as elements of $\pi_0(\text{Diff}^+)$.

The same clearly applies to fibres of singularities that are adjacent to (A_3) . What singularities are adjacent to A_3 ? Let us give a family of examples.

Lemma 2.7.13. *Let $\mathcal{M} \subset \mathbb{C}[z_1, z_2]$ be the maximal ideal, generated by z_1 and z_2 , and $f(z_1, z_2)$ be a non-zero element of \mathcal{M}^4 . Then f is adjacent to (A_3) .*

Proof. Consider $f_\beta(z_1, z_2) = f(z_1, z_2) + \beta z_1^2$, where $\beta > 0$ is an arbitrarily small constant. Considering Hessians, this is a corank-one singularity; the parametric Morse lemma (see e.g. [7, Section 1.3]) implies it is an (A_m) singularity, for some m . As (A_{k+1}) is adjacent to (A_k) , for all k , it is enough to show that $m \geq 3$. To do so, we estimate the Milnor number μ of f_β . The ideal \mathcal{I} generated by its partial derivatives is contained in

$$\mathcal{M}^3 + \langle z_1 \rangle = \langle z_1, z_2^3 \rangle \tag{2.7.18}$$

and thus

$$\mu = rk(\mathbb{C}[z_1, z_2]/\mathcal{I}) \geq rk(\mathbb{C}[z_1, z_2]/\langle z_1, z_2^3 \rangle) = 3. \tag{2.7.19}$$

□

This implies that the stabilizations of the polynomials $f \in \mathcal{M}^4$, of the form $f(z_1, z_2) + z_3^2 + \dots + z_{n+1}^2$, are adjacent to the (A_3) singularity in the appropriate dimension.

Remark 2.7.14. *The reader might wonder whether one could find two Lagrangian spheres L, L' in an odd complex-dimensional (A_2) fibre with $[L] \cdot [L'] = 0$, despite Lemma 2.7.12, as one could then imitate the Whitney trick argument above to smoothly isotope them apart. Suppose L and L' are two such Lagrangian spheres. Considering the intersection matrix on homology, we must have $k[L] = m[L']$, some integers k, m . By [3, Corollary 1.5], we have $k, m = \pm 1$. Thus L and L' are homologous, and Lemma 2.7.12 and the remark thereafter apply.*

Chapter 3

Lagrangian tori in four-dimensional Milnor fibres

3.1 Background on singularity theory

3.1.1 Basic properties

Unless otherwise specified, the singularity theory facts stated here are taken from the excellent survey [7].

Definition 3.1.1. *Let $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ be a holomorphic function such that $f(0) = 0$, $df|_0 = 0$ and $df \neq 0$ on the punctured ball $B_r^*(0)$, some sufficiently small r . An isolated hypersurface singularity at zero is the equivalence class of the germ of such an f , up to biholomorphic changes of coordinates (fixing 0).*

Whenever we use the term *singularity*, unless otherwise specified, we shall be referring to an isolated hypersurface singularity. We shall make repeated use of the following notion:

Definition 3.1.2. *Let (M, ω_M, θ_M) and (N, ω_N, θ_N) be exact symplectic manifolds. An exact symplectic embedding $M \hookrightarrow N$ is an embedding $f : M \rightarrow N$ such that*

$$f^*(\theta_N) = \theta_M + dh \tag{3.1.1}$$

where h is a compactly supported function on the interior of M . An exact symplectomorphism is defined similarly.

Milnor fibres: definition and independence of choices

Smooth category. Let $h : \mathbb{C}^{n+1} \rightarrow \mathbb{R}$ be the function given by

$$h(x_0, \dots, x_n) = |x_0|^2 + \dots + |x_{n+1}|^2. \tag{3.1.2}$$

Let f be a singularity. The restriction $h : f^{-1}(0) \rightarrow \mathbb{R}$ is a real algebraic function. By the Curve Selection Lemma ([51, Chap. 3]), it has isolated critical values. Let δ_f be the smallest positive one. For any $\delta < \delta_f$ and sufficiently small ϵ_δ , we have $f^{-1}(\epsilon_\delta) \cap S_{\sqrt{\delta}}(0)$, where $S_{\sqrt{\delta}}(0)$ is the sphere of radius $\sqrt{\delta}$ about the origin.

Definition 3.1.3. [51] *The Milnor fibre of f is the smooth manifold with boundary $f^{-1}(\epsilon_\delta) \cap \overline{B}_{\sqrt{\delta}}(0)$, for any such $\delta < \delta_f$ and $\epsilon_\delta \neq 0$. As a smooth manifold, this is independent of choices.*

Instead of h , consider any real algebraic function

$$\tilde{h} : \mathbb{C}^{n+1} \rightarrow [0, \infty) \quad (3.1.3)$$

such that $\tilde{h}^{-1}(0) = 0$. The Curve Selection Lemma still applies: the restriction of \tilde{h} to $f^{-1}(0)$ also has isolated critical values. For sufficiently small δ and ϵ_δ , $f^{-1}(\epsilon_\delta)$ is transverse to the δ -level set of \tilde{h} , and the manifolds

$$f^{-1}(\epsilon_\delta) \cap \{\tilde{h}(x_0, \dots, x_{n+1}) \leq \delta\} \quad (3.1.4)$$

are also copies of the Milnor fibre.

Symplectic category. The affine space \mathbb{C}^{n+1} comes with a ‘standard’ exact symplectic form, which is the usual Kaehler form on \mathbb{C}^{n+1} : $\omega = d\theta$, where $\theta = \frac{i}{4} \sum_i x_i d\bar{x}_i - \bar{x}_i dx_i$. This restricts to an exact symplectic form on any of the Milnor fibres. By construction, the associated negative Liouville flow is the gradient flow of $h(x)$ with respect to the standard Kaehler metric. Suppose we use the cut-off function

$$h_A(\mathbf{x}) = \|A\mathbf{x}\|^2 \quad (3.1.5)$$

for some $A \in GL_{n+1}(\mathbb{C})$. On \mathbb{C}^{n+1} , the negative gradient flow of h points strictly inwards at any point of the real hypersurface $\|A\mathbf{x}\|^2 = \delta$. (This is of course true for any δ , though we only need it for $\delta < \delta_{f,A}$.) In particular, for sufficiently small δ and ϵ_δ , the copy of the Milnor fibre given by

$$f^{-1}(\epsilon_\delta) \cap \{h_A(\mathbf{x}) \leq \delta\} \quad (3.1.6)$$

is an exact symplectic manifold with contact type boundary (i.e. a Liouville domain), and we can attach cylindrical ends to it using the Liouville flow on a collar neighbourhood of the boundary. We call this the *completed Milnor fibre* of f , and denote it M_f .

Independence of choices. To define M_f , we chose $A \in GL_{n+1}(\mathbb{C})$, then δ , then ϵ . We show below that M_f is independent of those choices (Lemma 3.1.4) and of the holomorphic representative of the singularity (Lemma 3.1.5).

Lemma 3.1.4. *Given a holomorphic function $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$, the completed Milnor fibre M_f is independent of the choice of A , δ and ϵ up to exact symplectomorphism.*

Proof. One can use symplectic parallel transport arguments, with varying choices of total spaces and bases. The key technical tool in all of these arguments is Lemma 9.4 of [40]. To understand independence of δ and ϵ in that framework, see [40, Section 9]. Let’s look at independence of A . Consider the map:

$$\pi : \mathbb{C}^{n+1} \times GL_{n+1}(\mathbb{C}) \rightarrow \mathbb{C} \times GL_{n+1}(\mathbb{C}) \quad (3.1.7)$$

$$(\mathbf{x}, A) \mapsto (f(\mathbf{x}), A) \quad (3.1.8)$$

where $GL_{n+1}(\mathbb{C}) \subset \mathbb{C}^{(n+1)^2}$ inherits the Kaehler structure. Fix a smooth path of matrices $A(t) \in GL_{n+1}(\mathbb{C})$, for $t \in [0, 1]$. Now choose a smooth family $\delta(t)$ with $0 < \delta(t) < \delta_{f,A(t)}$. We can find a smooth family of sufficiently small $\epsilon(t)$ such that

$$f^{-1}(\epsilon(t)) \cap \{h_{A(t)}(\mathbf{x}) = \delta(t)\} \quad (3.1.9)$$

and for each t , the space

$$f^{-1}(\epsilon(t)) \cap \{h_{A(t)}(\mathbf{x}) \leq \delta(t)\} \quad (3.1.10)$$

is a copy of the Milnor fibre of f . Consider the path $\gamma(t) = (\epsilon(t), A(t))$ in the base. Choose an open neighbourhood U of that path, such that all the fibres above points of U are smooth. We already have smooth choices of cutoffs for the fibres above $\gamma(t)$; extend this smoothly to all fibres above U . Call the resulting total space (that is, the union of truncated fibres) E . Lemma 9.3 of [40] allows us to modify the symplectic form on E so that on some collar neighbourhood of the boundary, horizontal tangent spaces are invariant under the Liouville flow on each fibre. (Also, restricted to any fibre, the symplectic form is unchanged.) We can now use symplectic parallel transport to get an exact symplectomorphism between any of the completed fibres above points of U . \square

Lemma 3.1.5. *The completed Milnor fibre M_f does not depend on the choice of holomorphic representative of f . More precisely, suppose $f = g \circ \rho$, some holomorphic change of coordinates ρ . Then there is an exact symplectomorphism from M_f to M_g .*

Proof. We'll use symplectic parallel transport again, and proceed in two steps. First, assume that ρ is linear: $\rho(\mathbf{x}) = D\rho|_0(\mathbf{x})$, with $D\rho|_0 \in GL_{n+1}(\mathbb{C})$. Consider the map:

$$\sigma : \mathbb{C}^{n+1} \times GL_{n+1} \rightarrow \mathbb{C} \times GL_{n+1}(\mathbb{C}) \quad (3.1.11)$$

$$(\mathbf{x}, A) \mapsto (f(A\mathbf{x}), A). \quad (3.1.12)$$

Pick a path $A(t) \in GL_{n+1}(\mathbb{C})$, with $t \in [0, 1]$, such that $A(0) = Id$, $A(1) = D\rho|_0$. Suppose that δ and ϵ are such that

$$f^{-1}(\epsilon) \cap \{\|\mathbf{x}\|^2 = \delta\} \quad (3.1.13)$$

and the space $f^{-1}(\epsilon) \cap \{\|\mathbf{x}\|^2 \leq \delta\}$ is a copy of the Milnor fibre of f . Then for all t , we have that

$$(f \circ A(t))^{-1}(\epsilon) \cap \{\|A(t)\mathbf{x}\|^2 = \delta\} \quad (3.1.14)$$

and, moreover, the space

$$(f \circ A(t))^{-1}(\epsilon) \cap \{\|A(t)\mathbf{x}\|^2 \leq \delta\} \quad (3.1.15)$$

is a copy of the Milnor fibre of $f \circ A(t)$, which we know to be smoothly isomorphic to a Milnor fibre of f . This gives smooth choices of cutoffs along the path $(\epsilon, A(t))$ in the base of σ . Proceeding similarly to the previous Lemma, we see that the completed Milnor fibres of f and $f \circ D\rho|_0$ are exact symplectomorphic.

We're left with the case of a general ρ . Consider the following map:

$$\tau : \mathbb{C}^{n+1} \times B_2(0) \rightarrow \mathbb{C} \times B_2(0) \quad (3.1.16)$$

$$(\mathbf{x}, c) \mapsto (f \circ (D\rho|_0 + c(\rho - D\rho|_0))(\mathbf{x}), c) \quad (3.1.17)$$

where $B_2(0) \subset \mathbb{C}$ is the disc of radius 2. There exists a (sufficiently small) $\delta > 0$ such that

$$(f \circ (D\rho|_0 + c(\rho - D\rho|_0)))^{-1}(0) \cap \{ \|D\rho|_0 \mathbf{x}\|^2 = \delta \} \quad (3.1.18)$$

for all $c \in [0, 1]$. This allows us find a path between copies of the Milnor fibres of $f \circ \rho$ and $f \circ D\rho|_0$, with smooth choices of cutoffs along those paths. We can then proceed with a symplectic parallel transport construction as before. \square

As much as possible, we have tried to refer to as the exact manifold with cylindrical ends as the *completed* Milnor fibre of f , and to the manifolds with boundary simply as copies of the Milnor fibre of f . (The reader should be aware that there is little practical difference for our purposes – in particular, we will only be considering exact compact Lagrangians.) Finally, it will be useful to note the following:

Remark 3.1.6. *Suppose that the function f has a single isolated singularity at 0, and is weighted homogeneous. One can check, for instance by writing down explicit maps, that the completed Milnor fibre M_f is exact symplectomorphic to any of the hypersurfaces $f^{-1}(\epsilon)$, for any $\epsilon \in \mathbb{C}^*$, equipped with the standard Kaehler exact symplectic form.*

Milnor numbers, intersection forms and sufficient jets

Fix a singularity f . A generic small perturbation to a Morse function \tilde{f} is called a *Morsification* of f . It has a collection of non-degenerate critical points near 0. The number of these critical points is independent of the Morse perturbation, and finite. It is called the *Milnor number* μ of f . Moreover, M_f is homotopy-equivalent to a wedge of μ half-dimensional spheres ([51]).

One can equip the middle-homology of M_f , $H_n(M_f) \cong \mathbb{Z}^\mu$, with an *intersection form* (for instance, formally, using cohomology with compact support). When the fibre has real dimension four ($n = 2$), the form is symmetric, and for any compact Lagrangian L , we have

$$L \cdot L = -\chi(L). \quad (3.1.19)$$

One can obtain a singularity in $n+2$ variables from a singularity in $n+1$ variables by adding a z_{n+2}^2 term, a process known as *stabilization*. This changes the intersection form; the resulting sequence of intersection forms has period four.

Milnor numbers have another important application, as follows. Consider the polynomial expansion of a singularity f at the singular point 0. The k -jet of f is called *sufficient* if any two functions with that jet are equivalent (that is, there exist a biholomorphic change of coordinates between them).

Theorem 3.1.7. [66, 8] *The $(\mu + 1)$ -jet of a function at an isolated critical point with Milnor number μ is sufficient.*

In particular, all of the singularities we consider are equivalent to polynomials.

Modality

Suppose you have a Lie group G acting on a manifold \mathcal{M} , and $p \in \mathcal{M}$.

Definition 3.1.8. [7, Section 1.6] *The modality of p under the action of G is the least integer m such that a sufficiently small neighbourhood of p is covered by a finite number of m -parameter families of orbits.*

The group of biholomorphic coordinate changes $(\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$ acts on the space of holomorphic functions $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$. This induces an action on the k -jet space, for any fixed k .

Definition 3.1.9. *The modality of a singularity f is the modality of any of its k -jets, for $k \geq \mu(f) + 1$.*

Loosely, the reader can think of modality as the “number of complex parameters” of the family that the singularity belongs to. For example, a modality one singularity is

$$x^4 + y^4 + ax^2y^2 \tag{3.1.20}$$

where a is a complex parameter such that $a^2 \neq 4$, associated to the cross-ratio of four lines.

Modality is unchanged under stabilization.

Adjacency and embeddings

Suppose $[f]$ and $[g]$ are singularities, described as equivalence classes of germs. We say that $[f]$ is *adjacent* to $[g]$ if there exists an arbitrarily small perturbations p such that $[f + p] = [g]$. For instance, $[x^m]$ is adjacent to $[x^n]$ for all $m \geq n$. Symplectically, we have that:

Lemma 3.1.10. [40, Lemma 9.9] *Suppose that $[f]$ and $[g]$ are singularities such that $[f]$ is adjacent to $[g]$. Then there exists an exact symplectic embedding from a non-completed Milnor fibre of g into a completed Milnor fibre of f .*

In particular, using Liouville flows at both ends, we can find an exact symplectic embedding of any compact subset of the (completed) Milnor fibre of g into the Milnor fibre of f . We shall make use of the following:

Theorem 3.1.11. [21, Theorem 10.1] *Any positive modality singularity is adjacent to a modality one singularity.*

Singularities of modality zero and one

Modality zero singularities are also known as *simple* singularities. They are also sometimes known as *elliptic* singularities. Their intersection forms correspond to Dynkin diagrams of the form A_m , D_m and E_6 , E_7 and E_8 . They are characterized by the following property.

Theorem 3.1.12. [65] *For $n = 2 \pmod{4}$, simple singularities have a negative definite intersection form. Moreover, they are the only singularities whose intersection form is definite.*

Modality one singularities are also known as *unimodal*. They are the ones we shall be primarily concerned with. They are classified into three families [7]. Three-variable representatives are as follows.

- Three parabolic singularities:

$$\begin{cases} T_{3,3,3}: & x^3 + y^3 + z^3 + axyz & a^3 + 27 \neq 0 \\ T_{4,4,2}: & x^4 + y^4 + z^2 + axyz & a \in \mathbb{C} \text{ such that } a^2 - 9 \neq 0 \\ T_{6,3,2}: & x^6 + y^3 + z^2 + axyz & a^6 - 432 \neq 0 \end{cases}$$

- The hyperbolic series $T_{p,q,r}$:

$$x^p + y^q + z^r + axyz$$

where $a \in \mathbb{C}^*$, and $p, q, r \in \mathbb{N}$ are such that $1/p + 1/q + 1/r < 1$.

- 14 ‘exceptional’ singularities. These are the objects, for instance, of strange or Arnol’d duality. We shall not consider them further here.

Unimodal singularities of more variables are stabilizations of these; unimodal singularities of one or two variables, singularities which stabilize to one of these.

The singularity $T_{p,q,r}$ has Milnor number $p + q + r - 1$.

Lemma 3.1.13. *The Milnor fibre of each $T_{p,q,r}$ is independent of a .*

Proof. For the three parabolic singularities, which are weighted homogeneous, this is automatic (see Remark 3.1.6). For the other singularities, we want to use a symplectic parallel transport argument, with the map

$$\mathbb{C}^4 \rightarrow \mathbb{C}^2 \tag{3.1.21}$$

$$(x, y, z, a) \mapsto (x^p + y^q + z^r + axyz, a) \tag{3.1.22}$$

The singular locus $\mathcal{C} \subset \mathbb{C}^2$ is algebraic, so its complement is connected. In order to use symplectic parallel transport, one just needs to be able to make choices of cutoffs that vary smoothly over a path in the smooth locus. This follows from the following observation: suppose that for some $r > 0$,

$$\{x^p + y^q + z^r + axyz = 0\} \cap \{|x|^2 + |y|^2 + |z|^2 = r\}. \tag{3.1.23}$$

Then, for any complex constant $\xi \neq 0$, and choices of p^{th} , q^{th} and r^{th} roots for ξ , we have

$$\{x^p + y^q + z^r + \xi^{\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1} axyz = 0\} \cap \{|\xi|^{\frac{2}{p}} |x|^2 + |\xi|^{\frac{2}{q}} |y|^2 + |\xi|^{\frac{2}{r}} |z|^2 = r\}. \tag{3.1.24}$$

This gives choices of cut-off functions h_A , for diagonal matrices A . □

There are many alternative standard representatives for these singularities; notably, in the case of the form $T_{p,q,2}$, we have [9, 10]:

$$T_{4,4,2}: \quad x^4 + y^4 + z^2 + ax^2y^2 \quad a^2 \neq 4 \tag{3.1.25}$$

$$T_{6,3,2}: \quad x^6 + y^3 + z^2 + ax^2y^2 \quad 4a^3 + 27 \neq 0 \tag{3.1.26}$$

$$T_{p,q,2}: \quad x^p + y^q + z^2 + ax^2y^2 \quad a \neq 0 \quad (1/p + 1/q < 1/2) \tag{3.1.27}$$

(The attentive reader might wonder why the numbers of values of a that are excluded are different for the previous presentations; this is due to redundancies for some of the descriptions.)

Using Tougeron's result on sufficient jets (Theorem 3.1.7), we are free to add monomials of degree at least $p + q + 1$ to these. Thanks to classification results of Arnol'd [10, Section 14], one can actually do even more than that. We shall use the representation

$$T_{p,q,2} : (x^{p-2} - y^2)(x^2 - \lambda y^{q-2}) + z^2 \quad (3.1.28)$$

for any $\lambda \in \mathbb{C}$ such that the resulting polynomial has an isolated singularity.

Whenever $p' \geq p$, $q' \geq q$ and $r' \geq r$, the singularity $T_{p',q',r'}$ is adjacent to $T_{p,q,r}$. Moreover, each of the fourteen exceptional singularities is adjacent to a parabolic. An immediate consequence of Theorem 3.1.11 is the following.

Corollary 3.1.14. *Any positive modality singularity is adjacent to at least one of the parabolic singularities.*

For the intersection forms of parabolic and hyperbolic singularities, see Section 3.1.2.

3.1.2 Some Picard-Lefschetz theory

References for this section are the survey book of Arnol'd and collaborators [7] for classical material (in the smooth category), and Seidel's book [62] for a detailed exposition in the symplectic category. We use a small variation on his framework.

Technical note: All the symplectic parallel transport arguments posited here can be obtained, for instance, by adapting constructions used in [40, Section 9] and in the previous section. We shall not provide details here.

Fix a singularity f , and a Morsification \tilde{f} . Unless otherwise stated, Morsifications are assumed to have distinct critical values. Pick a regular value ϵ of \tilde{f} .

Lemma 3.1.15. *Let δ and ϵ_δ be as in Definition 3.1.3. Then provided the perturbation is sufficiently small, $\tilde{f}^{-1}(\epsilon_\delta)$ intersects $S_{\sqrt{\delta}}(0)$ transversely, $\tilde{f}^{-1}(\epsilon) \cap \overline{B}_{\sqrt{\delta}}(0)$ is a Liouville domain, and moreover, after attaching conical ends, this is exact symplectomorphic to the Milnor fibre of f . Call this space F_{ϵ_δ} .*

Proof. For instance, you can use symplectic parallel transport with a piece of the fibration $\mathbb{C}^{n+2} \rightarrow \mathbb{C}^2$ given by $(\mathbf{x}, \gamma) \mapsto (f(\mathbf{x}) + \gamma(\tilde{f}(\mathbf{x}) - f(\mathbf{x})), \gamma)$. \square

This discussion also holds with any of function h_A in lieu of the standard h . The map

$$\tilde{f} : \mathbb{C}^{n+1} \rightarrow \mathbb{C} \quad (3.1.29)$$

restricted to suitable open subsets of 0 in the domain and range, is a Lefschetz fibration. Here are some of its features.

Let F_{x_s} be the fibre above a singular point x_s . We'll assume that we attached conical ends in a coherent manner. We have a symplectic parallel transport map with the properties:

- Fix any two regular values ϵ_1 and ϵ_2 , and a smooth path γ between them avoiding the singular points. This determines exact symplectomorphisms between F_{ϵ_1} and F_{ϵ_2} .
- Fix any small regular value ϵ , a singular value x_s , and a smooth path between them that otherwise does not go through any singular value. This is called a *vanishing path*

for x_s . It determines a map from F_ϵ to F_{x_s} . The preimage of the singular point in F_{x_s} is a Lagrangian sphere, called the *vanishing cycle* associated to this singular value and path. The map on the fibres is an exact symplectomorphism from the complement of the vanishing cycle to the complement of the singular point.

- Modifying the above paths by a smooth isotopy (still avoiding the singular values) changes the resulting maps by pre-composing with an exact symplectic isotopy.

As well as a Lagrangian sphere in the fibre, a vanishing path gives a Lagrangian disk in the total space, by taking the union of all the vanishing cycles above that path. We will call such a Lagrangian disk a *Lefschetz thimble*.

Definition 3.1.16. Fix a regular value ϵ . A distinguished collection of vanishing paths is a cyclically ordered family of vanishing paths γ_i between ϵ and the singular values x_s , with $i = 1, \dots, \mu(f)$, such that:

- They only intersect at ϵ .
- Their starting directions $\mathbb{R}_+ \cdot \gamma'_i(0)$ are distinct.
- They are cyclically ordered by clockwise exiting angle at ϵ .

Remark 3.1.17. This differs slightly from tradition: it is more common to use an absolute ordering compatible with the one described above. This is necessary if the smooth value lies on the boundary of a domain. For the mutations we consider in the subsequent section, we only need the cyclic ordering.

The resulting vanishing cycles give an ordered, so-called *distinguished* basis for the middle-dimension homology of the Milnor fibre.

Remark 3.1.18. This story behaves well with respect to the parallel transport arguments of Lemmas 3.1.4, 3.1.5 and 3.1.10 – in particular, vanishing cycles get taken to vanishing cycles.

Dehn twists and mutations

Fix a distinguished collection of vanishing paths, say $\gamma_1, \dots, \gamma_\mu$. Pre-compose γ_i with a clockwise loop around γ_{i-1} . Call the resulting vanishing path $\tau_{i-1}(\gamma_i)$. See Figure 3-1. The

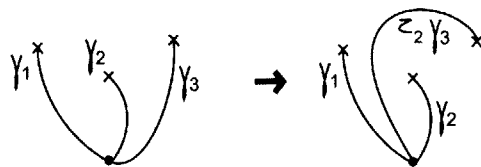


Figure 3-1: A mutation of vanishing paths

ordered collection of paths

$$\gamma_1, \dots, \gamma_{i-2}, \tau_{i-1}(\gamma_i), \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_\mu \tag{3.1.30}$$

Essentially local changes of the symplectic form

First, an observation:

Claim 3.1.21. *Let $\pi : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ be a holomorphic complex Morse function, with distinct critical values. Let ω be the usual Kaehler symplectic form on \mathbb{C}^{n+1} , and ω_b the one on the base \mathbb{C} . Let $c > 0$ be any positive constant. Then*

$$\omega' = \omega + c\pi^*\omega_b \quad (3.1.33)$$

is also a symplectic form. Moreover, restricted to any smooth fibre of π , it agrees with ω .

Proof. We need to check that ω' is non-degenerate at each point $x \in \mathbb{C}^{n+1}$. If x is a critical point of π , we simply have $\omega'_x = \omega_x$. Now suppose $x \in \mathbb{C}^{n+1}$ is distinct from the critical points. The two-form $c\pi^*\omega_b$ vanishes on the ‘vertical’ tangent space at x , and gives a symplectic form on the ‘horizontal’ tangent space determined by ω . In this case, the horizontal tangent space is just two-dimensional, and ω and $\pi^*\omega_b$ give area forms with the same sign, which implies the claim about ω' . \square

We shall later use the following technical result.

Lemma 3.1.22. *Suppose $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is a singularity, \tilde{f} a Morsification of it, and ϵ a regular value. Fix a vanishing path γ_i , and any compactly supported Hamiltonian isotopy of the Milnor fibre F_ϵ , say ϕ_t . Pick any open set U in the base, intersecting γ_i . Then we can modify the symplectic form to get a new symplectic form ω' such that:*

- $\omega' = \omega + c\tilde{f}^*\omega_b$ outside a compact set, where $c \geq 0$ is some constant, and ω_b the standard symplectic form on \mathbb{C} .
- $\omega'|_{\tilde{f}^{-1}(a)} = \omega|_{\tilde{f}^{-1}(a)}$ for all regular values a .
- The parallel transport on γ_i from ϵ to any point past U (using ω') is obtained by pre-composing the parallel transport for ω with ϕ_1 .

Proof. This is essentially Lemma 15.3 of [62]. Assume without loss of generality that U is a disc, and does not contain a singular value.

Let $U_{1/2}$ be a disc with same centre, and half radius. Let $H : F_\epsilon \rightarrow \mathbb{R}$ be the function whose Hamiltonian flow is ϕ_t . Using a suitable chosen smooth family of paths (including γ_i), this gives a Hamiltonian function on each fibre above U . Let t be the coordinate along γ_i , smoothly extended to U . Pick a bump function β on $U_{1/2}$ whose integral along γ_i is one. Let ω_b be the symplectic form on the base. For a sufficiently large constant k ,

$$\omega + d(\beta H \cdot dt) + k\tilde{f}^*\omega_b \quad (3.1.34)$$

is a symplectic form on $\tilde{f}^{-1}(U_{1/2})$. \square

The following is then immediate.

Corollary 3.1.23. *Changing ω in ways prescribed by the previous lemma, we can replace any vanishing cycle by a Hamiltonian isotopic Lagrangian sphere.*

Whenever we modify a symplectic form in such a way that $\omega' - \omega$ is exact, and $\omega' - \omega = c\pi^*\omega_b$ outside a compact set, we shall refer to the process as an *essentially local change of the symplectic form*. We shall want to modify the almost-complex structure accordingly.

Lemma 3.1.24. *Suppose ω' is obtained from ω as in lemma 3.1.22, and that J is an ω -compatible almost complex structure such that $\pi := \tilde{f}$ is (J, i) -holomorphic. Then we can find an ω' -compatible almost complex structure J' such that:*

- J and J' agree outside a compact set.
- J and J' agree when restricted to any fibre of π .
- π is also (J', i) -holomorphic.

Proof. Decompose the tangent space at any point x in the total space M using symplectic orthogonal complements:

$$T_x M = (\ker \pi) \oplus (\ker \pi)^{\perp \omega'}. \quad (3.1.35)$$

With respect to such a basis for this decomposition, J' must be of the form

$$J' = \begin{pmatrix} * & 0 \\ 0 & i \end{pmatrix} \quad (3.1.36)$$

and as J and J' agree on fibres, $*$ is uniquely determined. Moreover, the structure J' thus defined has the required properties. \square

One can also modify the symplectic form so that it becomes a ‘product’ in the neighbourhood of a fixed fibre. More precisely, we have the following:

Lemma 3.1.25. *Let $\pi : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ be a holomorphic complex Morse function, with distinct critical values. Let p be a regular value of π , Σ_p the fibre above p , and $B_r(p) \subset \mathbb{C}$ an open ball of radius r about p , whose closure does not contain any critical values. As before, let ω be the usual Kaehler form on \mathbb{C}^{n+1} , and ω_b be the one on the base. Let ϕ be the diffeomorphism*

$$\phi : \pi^{-1}(B_r(p)) \rightarrow \Sigma_p \times B_r(p) \quad (3.1.37)$$

given by using symplectic parallel transport with respect to ω along straight-line segments starting at p . (Note ϕ is compatible with projection to $B_r(p)$.) Fix a compact subset $K \subset \Sigma_p$. We claim that we can find a symplectic form ω' on \mathbb{C}^{n+1} such that:

- on $\phi^{-1}(K \times B_{r/2}(p))$, ω' is a ‘product’ symplectic form:

$$\omega' = \phi^*((\omega|_{\Sigma_p}, c\omega_b)) \quad (3.1.38)$$

for some constant $c > 0$;

- $\omega' = \omega + c\pi^*\omega_b$ outside a compact set;
- ω' and ω agree when restricted to any smooth fibre of π .

Proof. Let $\theta_p \in \Omega^1(\Sigma_p)$ be the restriction of the standard one-form θ to Σ_p . Fix $q \in B_r(p)$, and let Σ_q and θ_q be defined analogously. Let $\phi_q : \Sigma_q \rightarrow \Sigma_p$ be the result of symplectic parallel transport along a straight-line segment. This has the feature that

$$\phi_q^*(\theta_p) = \theta_q + d\rho_q \quad (3.1.39)$$

for some compactly supported smooth function ρ_q on Σ_q . Varying over q , these give a smooth function on $\pi^{-1}(B_r(p))$, with bounded support, say ρ .

Let ψ be a bump function on \mathbb{C}^{n+1} with bounded support inside $\pi^{-1}(B_r(p))$, and such that $\psi = 1$ on $\phi^{-1}(K \times B_{r/2}(p))$. Let $\omega^{pr} = \phi^*((\omega|_{\Sigma_p}, 0))$ be a degenerate ‘product’ form on $\pi^{-1}(B_r(p))$. We have $\omega^{pr} = d\theta^{pr}$, where θ^{pr} is defined to be:

$$\theta^{pr} := \phi^*((\theta|_{\Sigma_p}, 0)) - d\rho. \quad (3.1.40)$$

Now set

$$\omega' = \omega + d(\psi(\theta^{pr} - \theta)) + c\pi^*\omega_b \quad (3.1.41)$$

for some constant c to be determined. This is certainly closed. As $\theta^{pr} - \theta$ vanishes fibre-wise, ω' agrees with ω when restricted to any smooth fibre. Thus we just need to pick a constant c large enough to ensure that the form is non-degenerate at each point. \square

Analogously to Lemma 3.1.24, one can show the following.

Lemma 3.1.26. *Suppose ω' is obtained from ω as in Lemma 3.1.25, and that J is an ω -compatible almost complex structure such that π is (J, i) -holomorphic. Then we can find an ω' -compatible almost complex structure J' such that:*

- J and J' agree outside a compact set.
- J and J' agree when restricted to any fibre of π .
- π is also (J', i) -holomorphic.

These requirements determine J' uniquely. Moreover, on the set $\phi^{-1}(K \times B_{r/2}(p))$, on which ω' is a product, J' will correspond to the product $(J|_{\Sigma_p}, i)$.

Remarks on more general Lefschetz fibrations

Most of the features described previously in Section 3.1.2 are also found in more general Lefschetz fibrations (we shall only be concerned with some whose total spaces are complex hypersurfaces, or open subsets thereof). One can for instance consult [62, Chapter III]. In particular, we will use the concepts of symplectic parallel transport, distinguished collections of vanishing paths and vanishing cycles, and Lefschetz thimbles in such settings. They are defined completely analogously, and their properties are similar, with the caveat that vanishing cycles may not generate the homology of a smooth fibre, and that two different singular values may give the same vanishing cycle. One can still perform essentially local changes to the symplectic form to realise Hamiltonian isotopies of vanishing cycles, with prescriptions analogous to Lemmas 3.1.22 or 3.1.25.

3.1.3 Milnor fibres of functions of two variables, following A’Campo

Given a singularity of two variables, A’Campo [5] presents a way of describing the Milnor fibre of that singularity, together with favourite vanishing cycles and paths. This uses certain totally real deformations of the singularity; such deformations had earlier been considered by A’Campo [4] and independently Gusein-Zade [33] to calculate intersection forms of singularities. We give a brief outline here, while highlighting some features we shall use.

Divides and real deformations

Definition 3.1.27. [4] *Let R be the disjoint union of r copies of $[0, 1]$, and $D_\epsilon \subset \mathbb{R}^2$ the disc of radius ϵ centred at 0 . An r -branched divide (‘partage’) of D_ϵ is an immersion $\alpha : R \rightarrow D_\epsilon$ such that:*

- $\alpha(\partial R) \subset \partial D_\epsilon$, $\alpha(\overset{\circ}{R}) \subset \overset{\circ}{D}_\epsilon$, and $\alpha(R)$ is connected.
- $\alpha(J)$ only has ordinary double points.
- A region is a connected component of $D_\epsilon \setminus \alpha(R)$ that does not intersect the boundary. For any two regions A and B , we have $\bar{A} \cap \bar{B} = \alpha(I)$, where $I \subset R$ is a connected segment (possibly empty, or a point).

A’Campo considers real deformations of certain singularities whose zero-loci give divides.

Proposition 3.1.28. [4, Théorème 1] *Let $f(x, y)$ be a polynomial such that*

- $f(0) = 0$, and f has an isolated at the origin.
- f decomposes into a product of irreducible factors at 0 (over \mathbb{C}), each of which are polynomials with real coefficients.

Then f has a real polynomial deformations $f(x, y; t)$, $t \in \mathbb{R}$, such that $f(x, y; 0) = f(x, y)$ and for all sufficiently small $t \neq 0$, we have that:

- The real curve $C_t = \{(x, y) \mid f(x, y; t) = 0\}$ is an r -branched divide.
- The number k of double points of C_t satisfies $2k - r + 1 = \mu$, where μ is the Milnor number of f at zero.

Using that fact that C_t is a divide, the second condition is equivalent to:

$$\# (\text{regions of the divide } C_t) + k = \mu. \quad (3.1.42)$$

What is the significance of this? Let $\tilde{f}(x, y) := f(x, y; 1)$. Consider the zero-locus of \tilde{f} in the real $x - y$ plane. Each crossing corresponds to a saddle-type critical value for *real* variables, and thus, also, to a critical value for \tilde{f} as a function of complex variables. In each region of the divide, the real function \tilde{f} must attain at least one maximum or minimum. Similarly, these extrema also give critical values of the complex function. Thus to each region corresponds at least one critical point of \tilde{f} . As \tilde{f} is a divide, it must be that each region actually corresponds to exactly one critical point of \tilde{f} . Informally speaking, this real deformation of f ‘sees’ a full Milnor-number’s worth of critical points. We shall call such deformations *good real deformations*.

Remark 3.1.29. While \tilde{f} is non-degenerate, it does not in general have pairwise distinct critical values. If needed, one can remedy this by making a small further perturbation.

Remark 3.1.30. The reader might be concerned about the second assumption in Proposition 3.1.28. However, A'Campo notes that for the purposes of understanding the topological type of a singularity (including the construction of Milnor fibres below), it is not actually restrictive. For instance, one can use work of Lê Dũng Tráng and Ramanujam [47], who show that within a smooth one-parameter family of isolated hypersurfaces singularities in two variables with constant Milnor numbers, the topological type of the singularity does not change.

Associating Milnor fibres to divides

Consider the polynomial

$$f(x, y) = xy(x - y)(x + y). \quad (3.1.43)$$

A good real deformation is given by:

$$f(x, y; t) = xy(x - y + 2t)(x + y + t). \quad (3.1.44)$$

Let $\tilde{f}(x, y) = f(x, y; 1)$. The associated divide is given in Figure 3-3. Given a divide, A'Campo

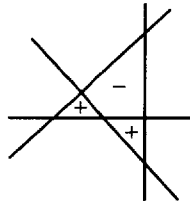


Figure 3-3: A divide for the product of four linear functions

algorithmically associates to it an oriented (topological) Riemann surface, which is the Milnor fibre of f . Consider the divide as a planar graph, and proceed as follows:

- To each edge of the graph, associate a ribbon-like strip with one half twist.
- Replace each of the intersections of the graph by a cylinder, attached to each of the four incoming ribbons.

The attachments are made as in Figure 3-4, which continues our example. Note that changing the direction of a half twist alters the embedding into \mathbb{R}^3 , but not the actual surface.

Pick a regular value of the form $-i\eta$, where $\eta \in \mathbb{R}_+$ is sufficiently small. As vanishing paths, take straight lines between $-i\eta$ and the critical values. (To be rigorous, you need to pick a very small Morse perturbation of \tilde{f} such that all the critical values are distinct, and, w.l.o.g., real. It will turn out that it does not matter what the resulting orderings are within any of the types of critical values: it only matter that all the minima be first, then saddles, then maxima.) For these choices, vanishing cycles are given as follows:

- The cycles corresponding to real saddles are given by meridional curves of the cylinders introduced for the associated double points.

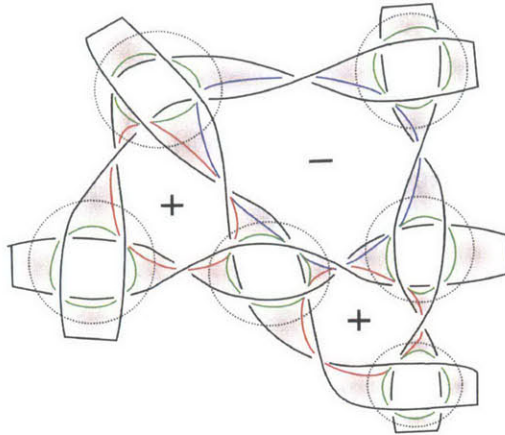


Figure 3-4: Milnor fibre of $x^4 + y^4$. Cylinders associated to crossings are inside the dotted circles. The vanishing cycles corresponding to the minimum, saddles, and maxima, are, respectively, in blue, green and red.

- The cycles corresponding to real extrema are given by going along the ribbons corresponding to the boundary of the region the extremum lies over.

(For our example, see Figure 3-4.)

Properties of A'Campo's construction

This description gives preferred orientations for each of the vanishing cycles: anticlockwise in the plane that the projection of the surface lives in (e.g., the paper). The Milnor fibre itself carries a natural orientation, as a complex curve.

Proposition 3.1.31. [4, p. 4] *Let V_1, \dots, V_μ be the vanishing cycles given by A'Campo's algorithm. We have the following intersection numbers:*

- $V_i \cdot V_j = 1$ if V_i corresponds to a region with a maximum, and V_j to a saddle in the boundary of that region.
- $V_i \cdot V_j = -1$ if V_i corresponds to a region with a minimum, and V_j to a saddle in the boundary of that region.
- $V_i \cdot V_j = 1$ if V_i corresponds to a maximum, V_j to a minimum, and the two regions share an edge.
- $V_i \cdot V_j = 0$ otherwise. (In particular, if V_i and V_j correspond to distinct critical points of the same real type.)

In practice, one often uses the preferred orientations of the vanishing cycles, together with the signs of the intersection numbers described above, to determine this orientation for the surface given by A'Campo's algorithm.

Now consider the Milnor fibre equipped with its exact symplectic structure. Examining A'Campo's argument, one notices the following.

Proposition 3.1.32. *After compactly supported Hamiltonian isotopies, we can arrange for the vanishing cycles given by A’Campo’s algorithm to intersect minimally. (In this case, in either zero or one point.)*

(It is now immediately clear that the ordering within each real–type class of singular points does not matter, as none of the corresponding cycles intersect – so one can trivially swap them using mutations.)

Remark 3.1.33. *While the algorithm described above is inherently restricted to functions of two variables, there exist some generalisations of A’Campo’s work in higher dimensions. For instance, one can related the flow category of a real Morse function with the directed Fukaya category of its complexification – see [37].*

3.1.4 Distinguished bases for $P(x_0, \dots, x_n) + x_{n+1}^{d+1}$

Background

Suppose you have two singularities, say $P(x)$ and $Q(y)$, with $x = (x_0, \dots, x_n)$ and $y = (y_0, \dots, y_m)$. We will want to use theorems that describe vanishing cycles for the joint singularity, $P(x) + Q(y)$, following Thom and Sebastiani [58]. Let μ be the Milnor number of P , and ν that of Q . Suppose you are given Morsifications \tilde{P} and \tilde{Q} . Let p_1, \dots, p_μ be the critical values of \tilde{P} , and q_1, \dots, q_ν the critical values of \tilde{Q} . Suppose also that you have chosen two regular values, say, respectively, p_* and q_* , and made choices of distinguished collections of vanishing paths between p_* and the p_i (respectively, q_* and the q_i). Gabrielov [30] describes how to use this data to construct a distinguished basis of vanishing paths and cycles for $P(x) + Q(y)$, in the smooth category. We shall later use such a construction, in the case that Q is simply a function of one variable, in the symplectic category. (When $Q(y) = y^2$, the process of adding Q is known as a *stabilization*; the Milnor number is unchanged, and the corresponding operation for vanishing cycles is well-known to experts; see e.g. [62, Chapter 3].) This is also considered by Futaki and Ueda [29], Section 2. For a more general case, with functions of arbitrarily many variables, the interested reader might consult [12, Section 6.3].

Assumptions and vanishing paths

Let \tilde{P} , \tilde{Q} , p_i and q_i be as in the introductory paragraph. Assume the p_i and q_i are distinct. Let γ_i be the vanishing path from p_* to p_i , and ζ_i be the vanishing path from q_* to q_i . We make the following (non-restrictive) assumptions:

Assumption 3.1.34. *There can find positive constants ϵ and r such that*

- $\gamma_i(t) \in B_\epsilon(p_*)$ for all i .
- $B_\epsilon(q_j)$ doesn’t intersect any of the ζ_k , for $k \neq j$.
- $p_* + q_* \in B_r(0)$, $p_i + q_i \in B_r(0)$ for all i and j , and there are no other critical values of $\tilde{P} + \tilde{Q}$ in $B_r(0)$.

The critical values of $\tilde{P} + \tilde{Q}$ are given by $\tau_{ij} = p_i + q_j$, and our assumption implies that they are distinct. Further deform the paths ζ_j such that:

- ζ_j only intersects the path $q_j + \gamma_i$ at q_j ;
- Ordered anti-clockwise, ζ_j arrives at q_j between the paths $q_j + \gamma_\mu$ and $q_j + \gamma_1$.

(See Figure 3-5.) We define a vanishing path between $p_* + q_*$ and $p_i + q_j$ as follows. First, concatenate the paths $p_* + \zeta_j$ (from $p_* + q_*$ to $p_* + q_j$) and $\gamma_i + q_j$ (from $p_* + q_j$ to $p_i + q_j$). Second, smooth out the results so that they only intersect at their starting points $p_* + q_*$. These form a distinguished collection of vanishing paths for $P + Q$. See Figure 3-5. We shall describe vanishing cycles for this collection below.

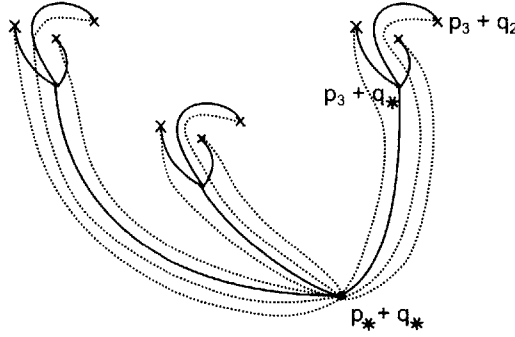


Figure 3-5: Vanishing paths for $P(x) + y^{d+1}$: concatenations of existing paths (full lines) and smoothings (dotted lines)

What if the original data did not satisfy the assumptions we made? Well, we can certainly deform it (in particular, scaling the perturbations of P and Q) until it does. Then take the vanishing paths described above, and deform them back.

Remark 3.1.35. *The ordering on the τ_{ij} is a lexicographic ordering induced by those on the q_j and p_i . While we usually assign a cyclic ordering to distinguished collections of paths, this construction actually requires an auxiliary absolute ordering for the p_i ; one can of course interpolate between different choices through series of mutations.*

The reader might also be wondering about the apparent lack of symmetry between the roles of the p_i and q_j . This is an artefact of presentation; in particular, it turns out that one can make a series of trivial mutations (the twists involved are between spheres that do not intersect) to get a distinguished basis of vanishing cycles with the other lexicographic ordering.

Matching paths and vanishing cycles

We now make the following assumptions on our choices of deformations (again, these are non-restrictive): that there is an open set $V \subset \mathbb{C}^{n+2}$ and $\delta > 0$ such that

- $V \subset B_R(0)$, some R such that $\{(x, y) \mid \tilde{P} + \tilde{Q} = p_* + q_*\} \cap B_R(0)$ is the Milnor fibre of $P + Q$.
- The map

$$\begin{aligned} \pi : \{(x, y) \mid \tilde{P} + \tilde{Q} = p_* + q_*\} \cap V &\rightarrow \mathbb{C} \\ (x, y) &\mapsto y \end{aligned}$$

is a Lefschetz fibration, with smooth fibre the Milnor fibre of P , exactly $d\mu$ critical points, and critical values y such that

$$p_i + \tilde{Q}(y) = q_* + p_*$$

some i , $1 \leq i \leq \mu$.

- Let the pre-images of q_* under Q be q_1, \dots, q_{d+1} . We'll assume that for each i and j , there is exactly one solution to $p_i + \tilde{Q}(y) = q_* + p_*$ inside $B_\delta(q_j)$. Call it p_{ij} .

For every i , the vanishing paths for the p_i determine vanishing paths from q_j to p_{ij} ; we may deform them so that they lie inside the balls $B_\delta(q_j)$. Consider the vanishing path ζ_j from q_* to q_j . Let \bar{q}_j be the critical point such that $\tilde{Q}(\bar{q}_j) = q_j$. Now ζ_j determines a (unique) path through \bar{q}_j between two of the pre-images of q_* , say q_{j_1} and q_{j_2} . Moreover, each pre-image of q_* lies at the end of at most two such paths, and any two paths intersect at most at one end-point. See Figure 3-6 for an example.

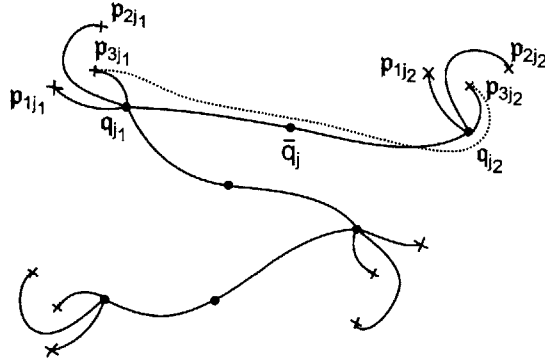


Figure 3-6: Singular values and matching paths for $\tilde{P} + \tilde{Q}$

Fix i and j , and consider the path from p_{ij_1} to p_{ij_2} obtained by concatenating the paths we have described from p_{ij_1} to q_{j_1} , q_{j_1} to q_{j_2} , and q_{j_2} to p_{ij_2} , and making a very small perturbation to get a smooth path. (See Figure 3-6; the smoothing should not intersect any of the other critical values.) By construction, this is a *matching cycle*: if you start with the fibre above a point in the interior of the path, the vanishing cycle obtained by deforming the fibre to p_{ij_1} is Hamiltonian isotopic to the vanishing cycle obtained by deforming to p_{ij_2} . After an essentially local change of the symplectic form (as in Corollary 3.1.23), can arrange for the vanishing cycles for both directions to agree exactly. One can then glue together the corresponding Lefschetz thimbles to obtain Lagrangian spheres in the total space. The sphere above the path between p_{ij_1} and p_{ij_2} is a vanishing cycle for $p_i + q_j$, with the vanishing path described in the previous subsection.

Remark 3.1.36. In [30], Gabrielov uses the projection to $\tilde{Q}(y)$ – in general, y consists of multiple variables – and constructs the vanishing cycle in $\{\tilde{P}(x) + \tilde{Q}(y) = p_* + q_*\}$ by using the product of the cycles for P and Q above a preferred path in the range of $Q(y)$. We only care about the case where y is a single variable, in which case a vanishing cycle for Q is just the union of two points. One can then check that the two descriptions are equivalent. In

particular, the equivalence between the two constructions makes it immediate to check that the paths we describe are indeed matching paths.

A cyclically symmetric scenario

In general, it might be complicated to study the branched cover $\tilde{Q} : \mathbb{C} \rightarrow \mathbb{C}$ to understand what the paths between q_{j_1} and q_{j_2} are. However, notice that all choices of Morsifications and vanishing cycles for Q are actually equivalent after an isotopy of the base. In particular, one can choose only to think about perturbations of Q of the form

$$\tilde{Q}(y) = y^{d+1} - ky \tag{3.1.45}$$

for some constant k , with $q_* = 0$, and, as vanishing paths, straight lines between the origin and the scaled roots of unity.

We shall use the following case: suppose that all of the p_i are real, $p_* = -i\eta$ (some $\eta \in \mathbb{R}_+$), and vanishing paths are given by straight line segments. (This is the scenario handed to us by A'Campo's construction.) Then

$$\{(x, y) \mid \tilde{P}(x) + y^{d+1} = c\} \cap B_{R_c} \tag{3.1.46}$$

is a copy of the Milnor fibre of $P + Q$, for any suitably chosen c and R_c . The projection to y has a cyclic symmetry of order $d + 1$, and we can arrange for singular values to be positive scalings of the $(d + 1)^{th}$ roots of unity. See Figure 3-7.

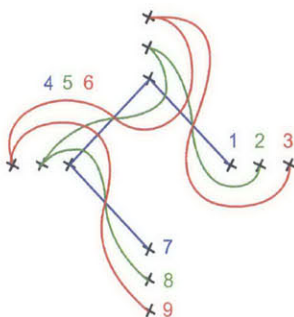


Figure 3-7: Cyclically symmetric scenario: matching paths for $P(x) + y^4$, and order of the corresponding vanishing cycles (case $\mu = 3$).

Deforming to $\{(x, y) \mid \tilde{P}(x) + y^{d+1} - ky = c'\}$ one can check that the paths drawn on that figure are matching paths, and give our distinguished collection of vanishing cycles.

Dehn twists revisited

Suppose you have a Lefschetz fibration with two matching paths which intersect *at one point*, away from the critical values. Let L and L' be the corresponding Lagrangian spheres. Then $\tau_L L'$ can also be described by a matching path; see Figure 3-8 for the local model (compare with Figure 3-1). Note that the change to the matching path for L' happens in an arbitrarily

small neighbourhood of the matching path for L . For further details and a proof, see [62, Chapter 3].

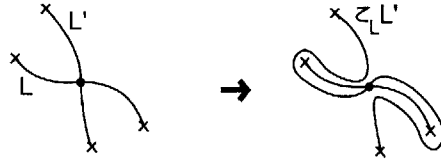


Figure 3-8: Dehn twist of a matching cycle by another: the paths intersect in their interior.

Suppose instead that the matching paths share a critical value (but do not otherwise intersect). Then $\tau_L L'$ can again be described by a matching path; see Figure 3-9.

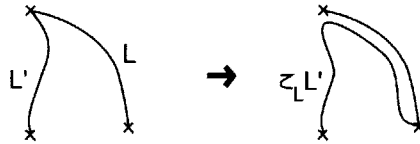


Figure 3-9: Dehn twist of a matching cycle by another: the paths intersect at an end.

3.1.5 Some extensions

Many of the results and techniques described in previous parts of this section extend to somewhat wider frameworks. We collect here those extensions that we shall later make use of. These will be used in Section 3.3.2 where we describe the Milnor fibre of $T_{3,3,3}$ with vanishing cycles, and in Section 3.3.3, which considers the general $T_{p,q,r}$.

Generalised Milnor fibres

Suppose we have a polynomial $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ with finitely many isolated critical points, and distinct critical values. Let $f^{-1}(c)$ be a smooth fibre. For all sufficiently large R , the sphere S_R intersects $f^{-1}(c)$ transversally (see e.g. [51, Lemma 2.8]). We shall call this the *generalised Milnor fibre* of f . (As with actual Milnor fibres, one could also attach conical ends to get an open exact symplectic manifold.) If f is a Morsification of the representative of a singularity with a single critical point, this is the same as the Milnor fibre. There's a Lefschetz fibration

$$f : V \subset \mathbb{C}^{n+1} \rightarrow D \subset \mathbb{C} \tag{3.1.47}$$

for suitably chosen open sets V and D (containing all the critical values), with fibre the generalised Milnor fibre. We shall consider such generalised Milnor fibres later, starting in Section 3.3.2. Among other things, they allow us to consider some functions that are not Morsifications of any singularity.

More on A'Campo's techniques

Suppose we have a polynomial f as above, and that $n = 1$. Moreover, suppose f is real, and has a real polynomial deformation $f(x, y; t)$, where $t \in \mathbb{R}$, with the same properties as good real deformations, as described in Proposition 3.1.28: $f(x, y; 0) = f(x, y)$, and for all sufficiently small $t \neq 0$, we have that:

- The real curve $C_t = \{(x, y) | f(x, y; t) = 0\}$ is an r -branched divide.
- The number k of double points of $C - t$ satisfies $2k - r + 1 = \mu$, where μ is the *total* number of critical points of a Morse perturbation of f .

Note that in most cases where we will use this, the zero locus of f will itself be a suitable r -branched divide, and one can just take the constant deformation. A'Campo's techniques extend to understanding how to construct a copy of the generalised Milnor fibre of f , together with preferred vanishing paths and corresponding vanishing cycles. The algorithm is the same as previously; one simply needs to check that [5] does not use the fact that the divide originated from a single isolated singularity.

Extending the Gabrielov construction to other Lefschetz fibrations

Suppose $U \subset \mathbb{C}^{n+1}$, and $g : U \rightarrow \mathbb{C}$ is a Lefschetz fibration with μ critical points. (We are not assuming that g is the Morsification of a singularity.) Let Q and \tilde{Q} be as in Section 3.1.4: for instance, $Q(y) = y^{d+1}$, and \tilde{Q} is a Morsification of Q . We can define a new Lefschetz fibration by taking

$$g + \tilde{Q} : U' \subset \mathbb{C}^{n+2} \rightarrow \mathbb{C} \tag{3.1.48}$$

$$(x, y) \mapsto g(x) + \tilde{Q}(y) \tag{3.1.49}$$

for a suitable open $U' \subset \mathbb{C}^{n+2}$, depending on U . This is a Lefschetz fibration with $d\mu$ critical points. There was nothing about Gabrielov's description that required g to be the Morsification of a singularity. In particular, the descriptions of Section 3.1.4 apply to this setting, giving us choices of vanishing paths, and explicit descriptions of the associated cycles, as matching cycles in an auxiliary fibration. (With care, it seems one could also apply the discussion to even wider collections of Lefschetz fibrations, though we will not need that.)

3.2 Background on Floer theory and Fukaya categories

3.2.1 What flavour of the Fukaya category do we use?

We will only consider real four-dimensional exact symplectic manifolds that are *Liouville domains* (compact, contact type boundary), or Liouville domains with an infinite cylindrical end attached to the contact boundary. Moreover, these will all have vanishing c_1 . (This is true of any Milnor fibre, and more generally of any smooth hypersurface.) We will use variations of the Fukaya category set up in Seidel's book [62].

$\mathbb{Z}/2$ -graded version

The objects of the Fukaya category, *Lagrangian branes*, consist of triples (L, \mathfrak{s}_L, E) where:

- L is a compact, exact, orientable Lagrangian surface (notice any such L will be spin);
- \mathfrak{s}_L is a spin structure on L ;
- E is a flat complex line bundle on L .

We will see after defining the A_∞ -maps that there are redundancies within these choices. As a preliminary, notice the following: Suppose L , as above, has genus g . Fix an ordered basis for $H_1(L)$ (recall our homology groups have \mathbb{Z} -coefficients unless otherwise specified). This determines an isomorphism between the space of flat complex line bundles on L and $(\mathbb{C}^*)^{2g}$, given by holonomies about oriented simple closed curves representing the basis. On the other hand, this choice of basis also determines an identification of the space of spin structures on L and $\{\pm 1\}^{2g}$: any spin structure on L is determined by its restriction to the simple closed curves, which can either be trivial (to which we associated 1) or non-trivial (-1). For more details, the reader could consult e.g. [38].

The difference with the set-up in [62] is that we are also using flat complex line bundles. We'll explain how to define Floer chain complexes and the A_∞ -structure in our case by twisting the maps in [62]. Suppose we have made universal choices of strip-like ends and consistent choices of regular Floer and perturbation data, as in [62]. Fix compact exact orientable Lagrangians L_0 and L_1 , with spin structures \mathfrak{s}_0 and \mathfrak{s}_1 . To keep notation simple, assume that $L_0 \pitchfork L_1$, and that no Hamiltonian perturbation is made for that pair. We take our coefficient field to be \mathbb{C} . Using Seidel's notation, we have

$$CF((L_0, \mathfrak{s}_0), (L_1, \mathfrak{s}_1)) = \bigoplus_{x \in L_0 \cap L_1} |o_x| \cdot \mathbb{C}.$$

Each of the pseudo-holomorphic discs u contributing to the differential give a map

$$d_u : |o_x| \cdot \mathbb{C} \rightarrow |o_y| \cdot \mathbb{C}$$

for some $x, y \in CF(L_0, L_1)$, and similarly for higher A_∞ -products. In the twisted case, we can use whatever regular data choices we have already made for the untwisted case. (In particular, we shall use the same pseudo-holomorphic discs to calculate differentials and A_∞ -operations.) Suppose L_0 and L_1 are further decorated with flat complex line bundles E_0 and E_1 . We define the Floer complex to be:

$$CF((L_0, \mathfrak{s}_0, E_0), (L_1, \mathfrak{s}_1, E_1)) = \bigoplus_{x \in L_0 \cap L_1} |o_x| \otimes \text{Hom}(E_0|_x, E_1|_x).$$

To define A_∞ -products, we need to use the parallel transport maps associated to the E_i :

$$\pi_i^{\partial u} : E_i|_x \mapsto E_i|_y$$

for $i = 0, 1$. The differential is given by:

$$d(|o_x| \otimes \phi) = \sum_u d_u |o_x| \otimes \pi_1^{\partial u} \circ \phi \circ (\pi_0^{\partial u})^{-1}.$$

The higher A_∞ -maps are defined analogously. To see that we do indeed get well-defined operations, and an A_∞ -category, one could go through the construction in [62] and decorate

the Banach bundles and operators that appear with the appropriate twists. Alternatively, notice the following: in the untwisted case, each of the A_∞ -relations, including $d^2 = 0$, holds homotopy class by homotopy class: each A_∞ -relation is obtained by considering the boundary of a moduli space of holomorphic discs with boundary conditions on a collection of Lagrangians, and there is a different component for each homotopy class of discs with such boundary conditions.

Finally, an observation: If you start with a brane (L, \mathfrak{s}_L, E) , modifying E by the action of an element of $\{\pm 1\}^{2g} \subset (\mathbb{C}^*)^{2g}$ is equivalent to keeping E fixed and modifying the spin structure. When carrying out computations later, we shall fix a basis for $H_1(L)$, assume that we are using the corresponding favourite spin structure (i.e. the one that restricts to the trivial spin structure for each curve in the basis; we will suppress it from the notation), and specify E by giving monodromy with respect to this basis.

Absolutely \mathbb{Z} -graded version

We are considering exact symplectic manifolds M with $c_1 = 0$. Choose a lift

$$LGr(M) \rightarrow \widetilde{LGr}(M) \tag{3.2.1}$$

where $LGr(M)$ is the Lagrangian Grassmanian of M , and $\widetilde{LGr}(M)$ its universal cover. If L is a Lagrangian with vanishing Maslov class, then we can find a consistent lift of the tangent planes to L . We denote such a lift by \widetilde{L} .

Consider the category with objects given by triples $(\widetilde{L}, \mathfrak{s}_L, E)$, where:

- L is a compact, exact, orientable Lagrangian of Maslov class zero, and \widetilde{L} as above;
- \mathfrak{s}_L is a spin structure on L ;
- E is a flat complex line bundle on L .

Again, one can define morphisms spaces and A_∞ -operations by starting with the set-up of [62], and twisting morphisms spaces and A_∞ -operations by the contributions of the flat line bundles. We get an A_∞ -category with an absolute \mathbb{Z} -grading, which recovers the \mathbb{Z}_2 -grading of the previous section.

3.2.2 Generation and split-generation

The Fukaya category $\mathcal{Fuk}(M)$ with either gradings is an A_∞ category. One can extend this to the category of twisted complexes, $Tw\mathcal{Fuk}(M)$; see e.g [45] for a detailed account. We shall use the following two definitions:

Definition 3.2.1. *A collection of objects B_1, \dots, B_m in $\mathcal{Fuk}(M)$ are said to generate $\mathcal{Fuk}(M)$ if in $Tw\mathcal{Fuk}(M)$, every object of $\mathcal{Fuk}(M)$ is quasi-isomorphic to a twisted complex built from copies of B_1, \dots, B_m . This means that every object in $\mathcal{Fuk}(M)$ can be obtained from B_1, \dots, B_m by using iterated mapping cones, and, if gradings are involved, grading shifts.*

The collection B_1, \dots, B_m is said to split-generate $\mathcal{Fuk}(M)$ if every object of $\mathcal{Fuk}(M)$ is quasi-isomorphic to a direct summand of a twisted complex built from copies of B_1, \dots, B_m .

We shall use the following:

Lemma 3.2.2. *Suppose that $L \in \text{Ob } \mathcal{Fuk}(M)$ is such that*

- $HF(L, L) \neq 0$, and
- $HF(L, B_i) = 0$ for all $i = 1, \dots, m$.

Then the collection B_1, \dots, B_m cannot generate or even split-generate $\mathcal{Fuk}(M)$.

Seidel studied the Fukaya category of Milnor fibres of weighted homogeneous singularities.

Theorem 3.2.3. *[Seidel] Let f be a weighted homogeneous polynomial with weights w_1, \dots, w_d , such that*

$$1/w_1 + \dots + 1/w_d \neq 1.$$

Then the absolutely graded Fukaya category of the Milnor fibre of f is generated by vanishing cycles.

Proof. This follows from combining [60, Section 4c] and [62, Proposition 18.17]. The results are stated for the Fukaya category with objects Lagrangian branes consisting of an absolutely graded Lagrangian submanifold together with a *Pin* structure. (For our set-up, this corresponds to only allowing line bundles with monodromies of the form $\{\pm 1\}^{2g} \subset (\mathbb{C}^*)^{2g}$ – we only consider orientable Lagrangian surfaces, so we think about spin structures instead of *Pin* structures.) However, the proofs extend to the case of Lagrangian submanifolds decorated with flat complex line bundles. \square

3.2.3 Convexity arguments

In the descriptions of Section 3.3, we will usually present the Milnor fibre of $T_{p,q,r}$ as an open subset of a larger exact symplectic manifold. To calculate A_∞ -products between compact Lagrangians inside the Milnor fibre, we can use the following:

Lemma 3.2.4. *(Abouzaid, see e.g. [2] or [62, Lemma 7.5]) Suppose (N, ω_N, θ_N) is an arbitrary exact symplectic manifold, (U, ω_U, θ_U) an exact symplectic manifold of the same dimension with contact type boundary, and $\iota : U \hookrightarrow \text{int}(N)$ an exact symplectic embedding. Suppose that we also have an ω_N -compatible almost complex structure J , whose restriction to U is of contact type near ∂U . Let R be a compact connected Riemann surface with boundary. Suppose $u : R \rightarrow N$ a J -holomorphic map such that $u(\partial R) \subset \text{int}(U)$. Then we have that $u(R) \subset \text{int}(U)$ as well. Moreover, this only requires u to be U -holomorphic in a neighbourhood of ∂U .*

3.3 Vanishing cycles for $T_{p,q,r}$

3.3.1 Case of $T_{p,q,2}$

We shall take the following defining equation for $T_{p,q,2}$ (given in Section 3.1.1):

$$\tau_{p,q,2}(x, y, z) = (x^{p-2} - y^2)(x^2 - \lambda y^{q-2}) + z^2$$

where $\lambda \in \mathbb{C}$ is any constant such that the polynomial has an isolated singularity at zero. (In particular, we can choose $\lambda = 1$, except when $p = q = 4$, in which case $\lambda = 2$ works.)

We can choose a real deformation $m_p(x, y)$ of $x^{p-2} - y^2$ and a real deformation $n_q(x, y)$ of $x^2 - y^{q-2}$ such that

$$h_{p,q}(x, y) := m_p(x, y)n_q(x, y)$$

has only non-degenerate critical points, and is a good real deformation of $\sigma_{p,q}(x, y) := (x^{p-2} - y^2)(x^2 - y^{q-2})$, in the sense of section 3.1.3, with divide given by Figure 3-10. (This was also considered by Gabrielov – see [33, Figure 3].)

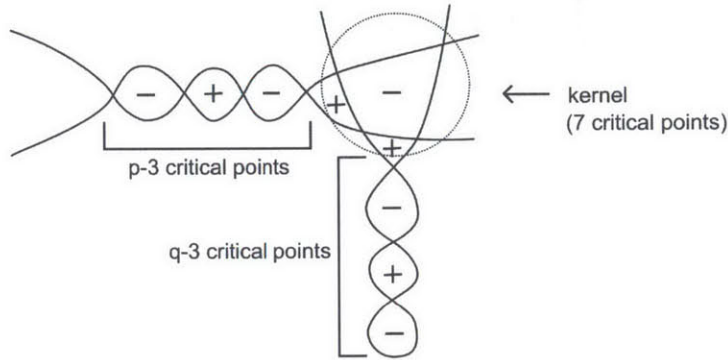


Figure 3-10: The divide associated to $h_{p,q}$. Whether each branch has one or two strands depends on the parity of p or q .

Regardless of the values of p and q , we can arrange for the two branches to intersect so as to give a fixed ‘kernel’ of seven critical points; the remainder of the critical points are arranged along two (A_m) -like chains.

Using A’Campo’s algorithm, we associate to this divide a copy of the Milnor fibre of σ , with vanishing paths and cycles. See Figure 3-11 for the case where $p = 4$ and $q = 5$. The seven coloured cycles (labelled a through g) correspond to the kernel; the three grey and black ones (labelled p_3 , q_3 and q_4), to the two chains of length $p - 3$ and $q - 3$. The orientation is as follows: the face closest to the reader is oriented anti-clockwise.

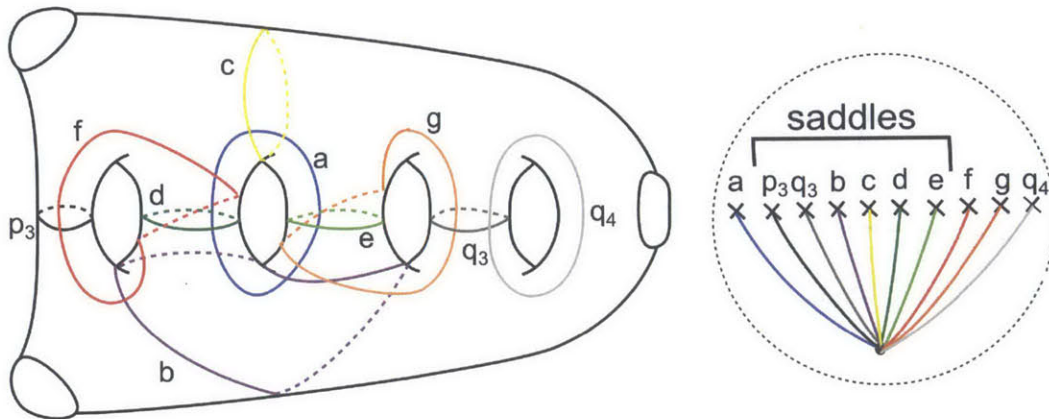


Figure 3-11: Milnor fibre of $\sigma_{4,5}$, vanishing paths and cycles.

Mutations on the Milnor fibre of $\sigma_{p,q}$

We perform mutations on the Milnor fibre of $\sigma_{p,q}$. As per Proposition 3.1.32, after an essentially local change to the symplectic form, we assume that to start with, the vanishing cycles intersect minimally. Moreover, after each mutation, we can also arrange for the new vanishing cycles to intersect minimally after Hamiltonian isotopy. (This is easy to check by hand as one performs the mutations, both in this section and in later ones, in which we find descriptions of more general $\mathcal{T}_{p,q,r}$. We will hereafter assume we do so.¹) Performing essentially local changes of the symplectic form as needed, we shall always assume that this is the case.

We shall include trivial mutations (involving vanishing cycles that do not intersect) for the case of $p = 4$ and $q = 5$. These are described in parenthesis. The reader might prefer to think of them as permissible re-orderings of the vanishing cycles. We make the following mutations.

$$f \mapsto (f^1 = \tau_e f) \mapsto f^2 = \tau_d f^1 \quad (3.3.1)$$

$$g \mapsto g^1 = \tau_e g \quad (3.3.2)$$

$$b \mapsto (b^1 = \tau_{q_3} b) \mapsto (b^2 = \tau_{p_3} b^1) \mapsto b^3 = \tau_a b^2 \mapsto (b^4 = \tau_{q_4} b^3) \mapsto b^5 = \tau_e b^4 \quad (3.3.3)$$

$$b^5 \mapsto (b^6 = \tau_{g^1} b^5) \mapsto b^7 = \tau_d b^6 \mapsto (b^8 = \tau_{f^2} b^7) \mapsto b^9 = \tau_c b^8 \quad (3.3.4)$$

Note that all the effective (that is, non-trivial) mutations just involve the seven cycles in the kernel. For different values of p and q , there will be trivial mutations following the same pattern as above.

Relabelling, the resulting collection of vanishing paths and cycles is given by Figure 3-12. We only keep track of the order of the vanishing paths, and not the exact trajectories. (After isotopy of the base, this is the only data that matters.) One can then make a sequence

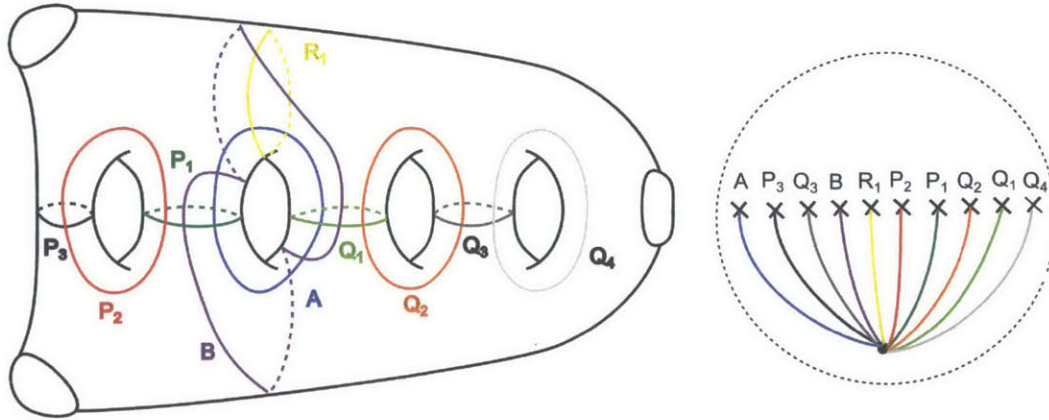


Figure 3-12: Milnor fibre of $\sigma_{4,5}$, vanishing paths and cycles after some mutations.

of trivial mutations, for instance as follows, to get the ordering of vanishing cycles given in Figure 3-13.

¹It seems one might hope to prove a more general result about vanishing cycles for a two-dimensional Milnor fibre and minimal intersections, perhaps using hyperbolic geometry.

$$Q_1 \mapsto \tau_{Q_4}^{-1} Q_1 \tag{3.3.5}$$

$$P_1 \mapsto \tau_{Q_4}^{-1} \tau_{Q_2}^{-1} P_1 \tag{3.3.6}$$

$$P_3 \mapsto \tau_{Q_2} \tau_{Q_4} \tau_{P_1} \tau_{Q_1} \tau_A P_3 \tag{3.3.7}$$

$$Q_3 \mapsto \tau_{P_1} \tau_{Q_1} \tau_A Q_3 \tag{3.3.8}$$

$$Q_4 \mapsto \tau_{P_1} \tau_{Q_1} \tau_A \tau_B \tau_{R_1} \tau_{P_2} \tau_{P_3} \tau_{Q_2} Q_4 \tag{3.3.9}$$

$$R_1 \mapsto \tau_{Q_1}^{-1} \tau_{P_1}^{-1} \tau_{Q_4}^{-1} \tau_{Q_3}^{-1} \tau_{Q_2}^{-1} \tau_{P_3}^{-1} \tau_{P_2}^{-1} R_1 \tag{3.3.10}$$

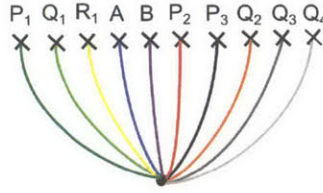


Figure 3-13: Vanishing paths for $\sigma_{4,5}$ after further mutations (all trivial).

For different values of p and q , one can proceed similarly. We label cycles as suggested by Figure 3-12. In general the indices of the P_i run from 1 to $p - 1$, and those of the Q_i from 1 to $q - 1$. For all of these cases, we have $r = 2$, and there is only R_1 .

Stabilization to $T_{p,q,2}$

We now use Section 3.1.4 to get a description of the vanishing cycles for $\tau_{p,q,2}(x, y, z) = \sigma_{p,q}(x, y) + z^2$. Use the Morsification of $\sigma_{p,q}$ to get a Morsification of $\tau_{p,q,2}$. Under the projection π of a Milnor fibre to the z coordinate, the resulting vanishing cycles are given by the matching paths of Figure 3-14. Their order is the same as the order for the vanishing

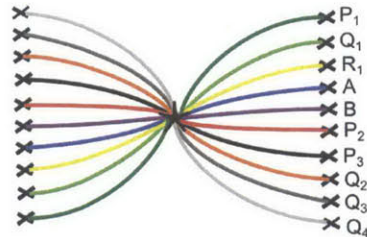


Figure 3-14: Matching paths for the Milnor fibre of $\tau_{4,5,2}$, after isotopy.

cycles of $\sigma_{p,q}$, and we shall use the same notation. After isotopy, we may assume that the matching paths all intersect in exactly one point, say \star . The fibre above that point, $\pi^{-1}(\star)$, is the Milnor fibre of $\sigma_{p,q}$, and each vanishing cycle restricts to the corresponding simple closed curve on $\pi^{-1}(\star)$ (as on Figure 3-12). In particular, our cycles precisely give the intersection form described by Gabrielov (Figure 3-2).

3.3.2 Case of $T_{3,3,3}$

Overview and strategy

The key is a one parameter family of polynomial maps $M(x, y, z; t)$, $t \in [0, 1]$, such that:

- $M(x, y, z; t) : \mathbb{C}^3 \rightarrow \mathbb{C}$ has distinct non degenerate critical points for all t .
- $M(x, y, z; t)$, for all $t \in [0, 1)$, has 14 critical points.
- $M(x, y, z; 1)$ is a deformation of $T_{3,3,3}$, and has eight critical points.
- $M(x, y, z; 0)$ is of the form

$$\tilde{m}(x, y) + Q(z)$$

where Q is a Morsification of z^3 , and $\tilde{m}(x, y)$ is a real polynomial with seven critical points, whose real zero locus gives a divide satisfying the conditions used in section 3.1.3.

We proceed in several steps. We start by combining the techniques of Sections 3.1.3 and 3.1.4 to give an explicit description of vanishing paths and cycles in the generalised Milnor fibre of $\tilde{m} + Q$; the vanishing cycles are given by matching paths in an auxiliary Lefschetz fibration.

For all $t < 1$, the generalised Milnor fibres of $M(x, y, z; t)$ are all exact symplectomorphic to one another. Moreover, we get an exact embedding of the Milnor fibre of $T_{3,3,3}$, say $\mathcal{T}_{3,3,3}$, into this space. In particular, the Floer cohomology (or more generally, higher A_∞ -products) between any two objects in the Fukaya category of $\mathcal{T}_{3,3,3}$ can be computed inside the generalised Milnor fibre of $\tilde{m} + Q$ instead.

We want to realise a distinguished collection of vanishing cycles for $T_{3,3,3}$ as a subset of a collection of distinguished vanishing cycles of the generalised Milnor fibre. (This is completely analogous to the process with Milnor fibres of two singularities, one of which is adjacent to the other.) Pick vanishing paths for $M(x, y, z; 0)$; as t increases, these deform to vanishing paths for $M(x, y, z; t)$, any $t < 1$. To get vanishing cycles with the desired property, we just need to pick paths that also deform to vanishing paths for $t = 1$ (that is, the paths for the eight critical values that remain at $t = 1$ do not get ‘broken’ as the six other critical values go off to infinity). We pick such paths.

We then work backwards: starting with the *known* configuration of vanishing cycles for $M(x, y, z; 0)$ (as matching paths in an auxiliary fibration), together with the corresponding vanishing paths, we make mutations to get to the configuration of vanishing paths that is compatible with the deformation in t . We track these mutations in the auxiliary fibration. This gives us a description of the vanishing cycles for $T_{3,3,3}$ as matching paths in the auxiliary Lefschetz fibration. Finally, we make a few further mutations, modelled on the case of $T_{p,q,2}$, to get to a nicer collection of matching paths, which is a basis giving Gabrielov’s intersection form (Figure 3-2).

A technical lemma

We shall make repeated use of the following lemma, which is really an observation:

Lemma 3.3.1. *Suppose that two matching paths intersect in a point (as in the left-hand side of Figure 3-15), and that after Hamiltonian isotopy, the corresponding vanishing cycles do*

not intersect in the fibre above that point. Then we can make a local change to one of the matching paths so that they no longer intersect, as in the right-hand side of Figure 3-15. The resulting matching cycle is Hamiltonian isotopic to the original one.



Figure 3-15: Local change of matching paths for vacuous intersections

Initial configuration

Consider the function

$$\tilde{m}(x, y) = 2((x + 0.25)^2 - 0.5(y + 0.25) - 2)(0.5(x + 0.25) + 2 - (y + 0.25)^2) \quad (3.3.11)$$

This has seven critical points. The real locus of $\{\tilde{m}(x, y) = 0\}$ is given in Figure 3-16. There

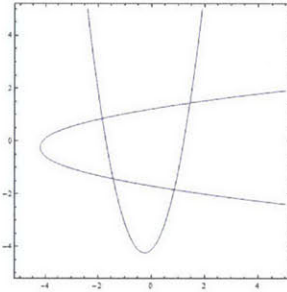


Figure 3-16: Real locus of $\{\tilde{m}(x, y) = 0\}$

are three critical values: a minimum, zero (multiplicity four) and a maximum (multiplicity two). Note that we've arranged for the minimum to be at $(0, 0)$. It realises the 'kernel' of the good real deformations we were considering for $T_{p,q,2}$, with the caveat that \tilde{m} is actually *not* the deformation of *any* isolated singularity. (One way to see this is to use the fact that the only singularities with Milnor number seven are A_7 , D_7 and E_7 ; the intersection form associated to the divide of \tilde{m} has non-trivial nullspace, so it is none of these.)

Consider a small deformation of \tilde{m} , say m . It defines a Lefschetz fibration

$$m : V \subset \mathbb{C}^2 \rightarrow \mathbb{C} \quad (3.3.12)$$

with seven critical points and values (V is a suitably chosen large open set). We can still use A'Campo's work to get the topological type of the fibre, and vanishing cycles and paths. The fibre is a twice-punctured genus three surface; if we choose, as for $T_{p,q,2}$, a regular value of the form $-i\eta$ ($\eta \in \mathbb{R}_+$), and vanishing paths given by straight lines, the vanishing cycles are precisely the coloured cycles in Figure 3-11 (labelled a through g).

Now let $Q(z) = z^3$, and let \tilde{Q} be a Morsification of Q . As in Section 3.1.4, we get a Lefschetz fibration

$$m + Q : V' \subset \mathbb{C}^3 \rightarrow \mathbb{C} \quad (3.3.13)$$

with 14 critical points and values (w.l.o.g. distinct). We can apply the symplectic version of Gabrielov's techniques to understand vanishing paths and cycles for this. To make calculations a little cleaner (with symmetry considerations), we will use the deformation of \tilde{m} given by:

$$\check{m}(x, y) = \tilde{m}(x, y) - 2xy. \quad (3.3.14)$$

This has seven critical points, and four critical values. One can check that in ascending real order, they correspond to a , then b and c , then d and e , then f and g . The minimum – corresponding to a – is still $(0, 0)$. Now consider the function

$$M(x, y, z) = \check{m}(x, y) + (18 + 8i)z^2 + \frac{16i}{3}z^3.$$

Critical values are given in Figure 3-17. Vanishing paths given by Gabrielov's algorithm are also on that figure. The yellow–and–purple paths (third and fourth) each give two vanishing paths, though of course the corresponding vanishing cycles do not intersect, so the order does not matter. The same goes for the green (fifth and sixth) and orange (seventh and eighth) paths. (Note: we have actually made trivial mutations to pick the other lexicographic ordering between vanishing cycles of m and Q , as it was graphically cleaner.)

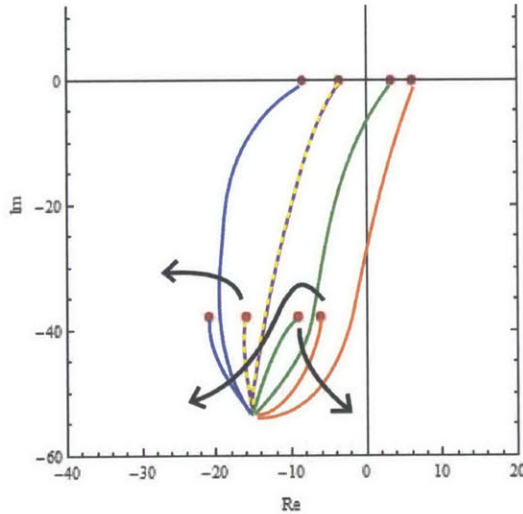


Figure 3-17: Critical values and vanishing paths for $M(x, y, z)$, with behaviour of the critical values under deformation.

We label the vanishing cycles in increasing order as $(a, 1), (a, 2), (b, 1), \dots, (g, 2)$, consistently with the letter labelling previously used for the kernel. Note $(a, 1)$ and $(a, 2)$ – both associated to minima of \check{m} – correspond to critical points of the form $(0, 0, \cdot)$. By construction, the critical points and values associated to $(a, 1)$ and $(a, 2)$ remain fixed as t varies.

Deformation to $T_{3,3,3}$

Consider the deformation

$$M(x, y, z; t) = \check{m}(x, y) + 2(3z + txy)^2 + 8iz^2 + \frac{16i}{3}z^3. \quad (3.3.15)$$

for $t \in [0, 1]$. For $t = 0$, this is simply $M(x, y, z)$. For $t = 1$, the $-2x^2y^2$ term in $\check{m}(x, y)$ is cancelled by $2t^2x^2y^2$; moreover, one can check that

$$M(x, y, z; 1) = x^3 + y^3 + 12xyz + 8iz^2 + \frac{16i}{3}z^3 + c_1x + c_2y + c_3xy \quad (3.3.16)$$

for some constants c_i , and that $M(x, y, z; 1)$ is a deformation of $x^3 + y^3 + z^3 + 12xyz$ (a representative of $T_{3,3,3}$); it has 8 critical points, and five critical values.

For each t , $M(x, y, z; t)$ has only non-degenerate critical points. As t increases from 0 to 1, six of the critical points of $M(x, y, z; t)$ go off to infinity; they correspond to three double critical values. The exit paths for the critical values are given in Figure 3-17; to check this is correct, the reader might want to use a computer software program. Mathematica code can be found in Appendix A.

Vanishing paths compatible with the deformation

We change the distinguished collection of vanishing paths of $M(x, y, z; 0)$ so as to have paths that deform to vanishing paths for $M(x, y, z; t)$ for all t . (As mentioned before, this just means the paths must, after isotopy, avoid the trajectories of the critical values that go off to infinity.) We choose the collection given by Figure 3-18. This can be obtained from the

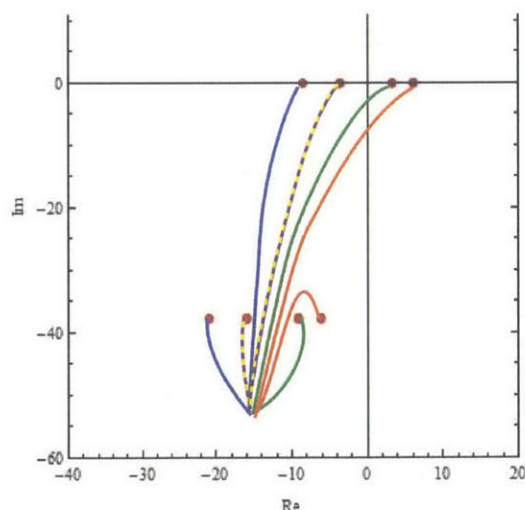


Figure 3-18: New distinguished collection of vanishing paths for $M(x, y, z; 0)$

previous distinguished collection through a series of mutations. The non-trivial ones are

$$(d, 2) \quad (\mapsto (d, 2)^1 = \tau_{(e,1)}(d, 2)) \quad \mapsto (d, 2)^2 = \tau_{(d,1)}(d, 2)' \quad (3.3.17)$$

$$(e, 2) \quad \mapsto (e, 2)^1 = \tau_{(e,1)}(e, 2) \quad (\mapsto (e, 2)^2 = \tau_{(d,1)}(e, 2)') \quad (3.3.18)$$

$$(f, 2) \quad (\mapsto (f, 2)^1 = \tau_{(g,1)}(f, 2)) \quad \mapsto (f, 2)^2 = \tau_{(f,1)}(f, 2)^1 \quad (3.3.19)$$

$$(g, 2) \quad \mapsto (g, 2)^1 = \tau_{(g,1)}(g, 2) \quad (\mapsto (g, 2)^2 = \tau_{(f,1)}(g, 2)^1) \quad (3.3.20)$$

$$(f, 2)^2 \quad (\mapsto (f, 2)^3 = \tau_{(e,1)}(f, 2)^2) \quad (\mapsto (f, 2)^4 = \tau_{(d,1)}(f, 2)^3) \quad (3.3.21)$$

$$(g, 2)^2 \quad (\mapsto (g, 2)^3 = \tau_{(e,1)}(g, 2)^2) \quad (\mapsto (g, 2)^4 = \tau_{(d,1)}(g, 2)^3) \quad (3.3.22)$$

$$(f, 1) \quad (\mapsto (f, 1)^1 = \tau_{(e,1)}(f, 1)) \quad \mapsto (f, 1)^2 = \tau_{(d,1)}(f, 1)^1 \quad (3.3.23)$$

$$(g, 1) \quad \mapsto (g, 1)^1 = \tau_{(e,1)}(g, 1) \quad (\mapsto (g, 1)^2 = \tau_{(d,1)}(g, 1)^1) \quad (3.3.24)$$

$$(a, 2) \quad (\mapsto (a, 2)^1 = \tau_{(b,1)}^{-1}(a, 2)) \quad (\mapsto (a, 2)^2 = \tau_{(c,1)}^{-1}(a, 2)^1) \quad (3.3.25)$$

The critical values that do not go to infinity are critical values for the Lefschetz fibration associated to $M(x, y, z; 1)$. We re-label the corresponding distinguished collection of vanishing paths, as follows (with order):

$$\square = (a, 1) \quad a = (a, 2) \quad b = (b, 2) \quad c = (c, 2) \quad d = (d, 2)^2 \quad e = (e, 2)^2 \quad f = (f, 2)^4 \quad g = (g, 2)^4$$

Vanishing cycles associated to the deformation-compatible vanishing paths

Starting with the configuration of matching cycles which we know to be associated to the collection of vanishing paths of Figure 3-17 (the ones we get using techniques from Section 3.1.4), we can perform the mutations of 3.3.2, and track the corresponding matching cycles. The result is in Figure 3-19. We have deleted the matching paths corresponding to critical values that go off to infinity.

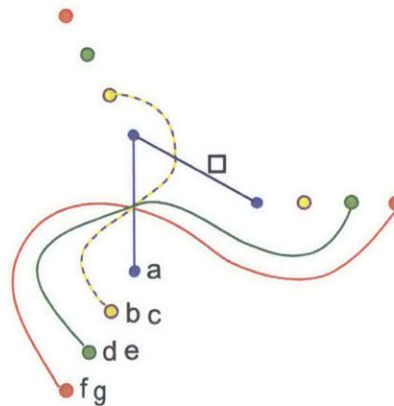


Figure 3-19: Matching paths for $T_{3,3,3}$. The colours codings are as before (note yellow/purple, green and orange each represent two paths), and dark blue corresponds to the cycle \square .

Further mutations

We perform further mutations on this distinguished basis for $T_{3,3,3}$, inspired by the mutations made earlier for $T_{p,q,2}$. They are as follows (as always, trivial mutations are in parenthesis):

$$f \mapsto (f^1 = \tau_e f) \mapsto f^2 = \tau_d f^1 \quad (3.3.26)$$

$$g \mapsto g^1 = \tau_e g \quad (3.3.27)$$

$$\square \mapsto (\square^1 = \tau_e \square) \mapsto (\square^2 = \tau_{g^1} \square^1) \mapsto (\square^3 = \tau_d \square^2) \mapsto (\square^4 = \tau_{f^2} \square^3) \quad (3.3.28)$$

$$\mapsto \square^5 = \tau_c \square^4 \quad (3.3.29)$$

$$b \mapsto b^1 = \tau_a b \mapsto b^2 = \tau_e b^1 \mapsto (b^3 = \tau_{g^1} b^2) \mapsto b^4 = \tau_d b^3 \mapsto (b^5 = \tau_{f^2} b^4) \quad (3.3.30)$$

$$\mapsto b^6 = \tau_c b^5 \quad (3.3.31)$$

These are exactly the mutations as for $T_{p,q,2}$, with the mutation of \square added in. Set

$$A = a \quad B = b^6 \quad R_2 = \square^5 \quad R_1 = c \quad P_2 = f^2 \quad P_1 = d \quad Q_2 = g^1 \quad Q_1 = e.$$

These are an ordered basis of vanishing cycles. After also making simplifying isotopies as in Lemma 3.3.1, the resulting configuration of matching cycles is given by Figure 3-20, where we have performed an isotopy of the base for further clarity. We can add trivial mutations (analogously to the end of Section 3.3.1) so that the order of the vanishing paths is that of the right-hand side of Figure 3-20. We can arrange for seven of the matching paths to intersect in

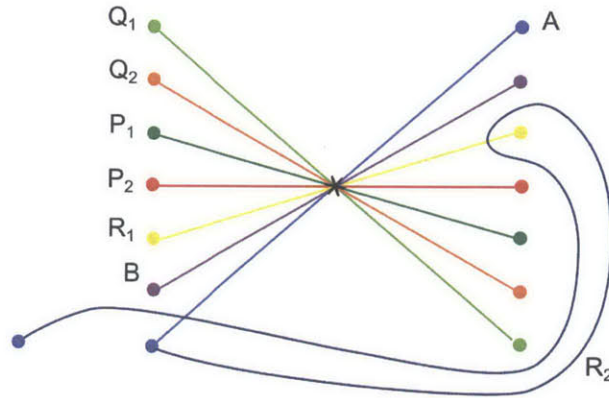


Figure 3-20: Matching paths for $T_{3,3,3}$, after mutations and isotopies.

exactly one point, \star . In the fibre above \star (a twice-puncture genus three Riemann surface), the vanishing cycles that are restrictions of the seven matching paths intersect precisely as in the $T_{p,q,2}$ case, i.e. Figure 3-11. One can track Dehn twists and check that R_1 and R_2 intersect exactly in one point. Similarly, A and R_2 intersect at two points, with opposite orientations. In particular, we have a configuration of vanishing paths recovering Gabrielov's presentation of the intersection form in Figure 3-2.

Isotopy to make A and R_2 disjoint

Lemma 3.3.2. *After Lagrangian isotopy, A and R_2 can be arranged to be disjoint (without affecting intersections with any of the other vanishing cycles).*

Proof. We start by describing a local model for the the intersections of A , R_1 and R_2 . Consider the hypersurface $\{(x, y, z) \mid x^3 + y^3 + z^2 = 1\}$. This is the Milnor fibre of D_4 . It can be viewed as the total space of a Lefschetz fibration in many different ways. Consider

$$\Pi_1 : \{(x, y, z) \mid x^3 + f(y, z) = 1\} \rightarrow \mathbb{C} \quad (3.3.32)$$

$$(x, y, z) \rightarrow x \quad (3.3.33)$$

where $f(y, z)$ is a Morsification of $y^3 + z^2$. There are six critical values; the generic fibre is a once-punctured torus. Matching paths corresponding to a distinguished collection of vanishing cycles for D_4 are given by Figure 3-21 (this is the collection given by Section 3.1.4 techniques after one mutation). Curves α , β and γ on the figure give a local model for the intersections of A , R_1 and R_2 . Now consider a different Lefschetz fibration, with, again, the

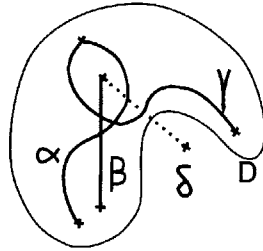


Figure 3-21: First Lefschetz fibration for a local model for the intersections of A (which corresponds to α), R_1 (β) and R_2 (γ).

Milnor fibre of D_4 (with three variables) as total space:

$$\Pi_2 : \{(x, y, z) \mid g(x, y) + z^2 = 1\} \rightarrow \mathbb{C} \quad (3.3.34)$$

$$(x, y, z) \rightarrow z \quad (3.3.35)$$

This has eight critical values; matching paths can be arranged to intersect in exactly one point. The fibre above that point is the Milnor fibre of $x^3 + y^3$ (D_4 with two variables); let's call it S . It is a three-punctured torus. Also, we know that through some (a priori unknown) sequence of mutations, we can get the same configuration of vanishing cycles as in Figure 3-21. (By "same", we mean that there is an exact symplectomorphism between the two representatives of the Milnor fibre that takes one ordered collection of vanishing cycles to the other, possibly after Lagrangian isotopies of some of the cycles.) The vanishing cycles corresponding to α , β , γ and δ are given by matching paths as in Figure 3-22, where they are labelled as α' , β' , γ' and δ' .

In particular, we can arrange for the matching paths for α', \dots, δ' to meet in exactly one point (w.l.o.g., $\Pi_2(S)$). The restrictions of the vanishing cycles α', \dots, δ' to S are themselves vanishing cycles for the singularity g . Consider the simple closed curves $\alpha'|_S, \dots, \delta'|_S$ on the three-punctured torus S . None can be contractible in the closure of S to a torus (this is true

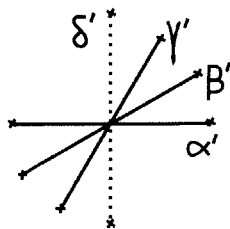


Figure 3-22: Second Lefschetz fibration for a local model for the intersections of A (which corresponds to α'), R_1 (β') and R_2 (γ').

of any vanishing cycle for g). By construction, the *signed* intersection number between $\alpha'|_S$ and $\gamma'|_S$ is zero. Thus they must be isotopic in the closure of S . (They are not isotopic in S itself: they are vanishing cycles for different critical values.) Hence there exists an isotopy of S such that $\alpha'|_S$ and the image of $\gamma'|_S$ are disjoint. As the symplectic area of any subset of S containing a puncture is infinite, there must be a compactly-supported *Hamiltonian* isotopy of S after which $\alpha'|_S$ and the image of $\gamma'|_S$ are disjoint. Extending this over the pre-image of a neighbourhood of $\Pi_2(S)$, we see that there is a compactly supported Hamiltonian isotopy such that α' and the image of γ' do not intersect. Thus the same is true of α and γ . Moreover, we can arrange for such an isotopy to have support in the preimage of the region D in Figure 3-21 (this equivalent to asking that none of the Lagrangian isotopic copies of γ intersect a fixed point of δ). In such a situation, the isotopy is contained to the local model. This implies that after a Hamiltonian isotopy of R_2 , A and R_2 can be made disjoint; the image of R_2 still intersects R_1 at one point. The other vanishing cycles lied away from this local model, so intersections with them are unchanged. \square

3.3.3 Case of a general $T_{p,q,r}$

Proposition 3.3.3. *Let $\mathcal{T}_{p,q,r}$ denote the Milnor fibre of $T_{p,q,r}$. Assume our subscripts are ordered so that $p \geq 3$, $q \geq 3$, and $r \geq 2$. The space $\mathcal{T}_{p,q,r}$ can be described as an open subset of the total space of a Lefschetz fibration π , whose fibre is the same as the Milnor fibre of $\sigma_{p,q}$ (a Riemann surface), and with $2p + 2q + r$ critical points and values. Of these critical points, $2(p + q + 1)$ of them correspond to critical points for the description of $\mathcal{T}_{p,q,2}$ as the total space of a Lefschetz fibration. There are $p + q + 1$ corresponding vanishing cycles, which are given by matching paths for that Lefschetz fibration; they all intersect in one point, \star .*

In the fibre above \star , which we shall call M_\star , the matching paths restrict to the configuration of vanishing cycles for $\sigma_{p,q}$ that we had already met (e.g. Figure 3-12).

As for the remaining $r - 2$ critical points, there is an A_{r-2} -type chain of matching paths between them; these give the remaining vanishing cycles for $\mathcal{T}_{p,q,r}$. The case of $p = 3$, $q = 4$ and $r = 5$ is given in Figure 3-23. For $r = 6$, the chain would be extended by the matching path for R_5 , which intersects only the matching path for R_4 , at their shared critical value, and so on.

Moreover, in all cases, there is a Hamiltonian isotopy such that the image of R_2 does not intersect A , and its intersections with other cycles are unchanged.

The reader might have noticed that for the cases $(p, q) = (3, 3)$, $(3, 4)$ and $(3, 5)$, there is

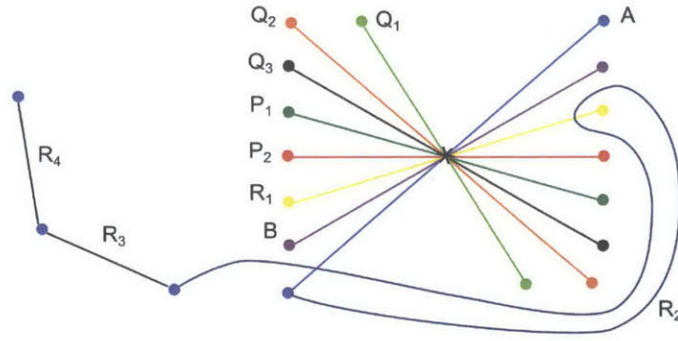


Figure 3-23: Matching paths giving a distinguished configuration of vanishing cycles for $T_{3,4,5}$; the labelling respects the previous ones, with analogous ordering.

no such singularity as $\sigma_{p,q}$. What do we mean by the “Milnor fibre” of $\sigma_{p,q}$ in this case? We take the generalized Milnor fibre of functions constructed so as to have analogous features. We already did such a thing for the case $(p,q) = (3,3)$: use the generalized Milnor fibre of $h_{3,3} := \frac{1}{2}\tilde{m}$ (see also Figure 3-16). For $(3,4)$, take the generalised Milnor fibre of $h_{3,4}$, a real deformation of $(x^2 - y^2)(x - y^2)$ with real zero locus given by Figure 3.3.3. Similarly, for $h_{3,5}$, use a real deformation of $(x^2 - y^3)(x - y^2)$ with real zero locus given by Figure 3.3.3. Notice that these fit into the pattern of the functions $h_{p,q}$ of Section 3.3.1. (Compare with Figure

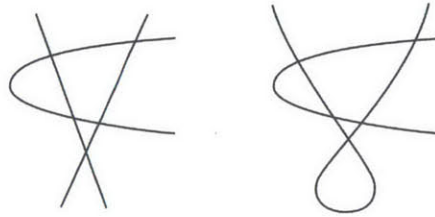


Figure 3-24: Real zero locus of $h_{3,4}$ (left) and $h_{3,5}$ (right).

3-10.) As with $\sigma_{p,q}$ and the functions $h_{p,q}$ of Section 3.3.1, A’Campo’s algorithm will give a Riemann surface with vanishing cycles consisting of seven simple closed curves – the ‘kernel’ introduced in Section 3.3.1 – and chains of the appropriate length attached to it.

Proof. For $T_{p,q,2}$ and $T_{3,3,3}$, this agrees with the description that we already have. We will first show that the theorem holds for $T_{3,3,r}$, with $r \geq 4$, by building on the work of the previous section (describing $T_{3,3,3}$). We will then combine this with what we know about $T_{p,q,2}$ to get the description for a general $T_{p,q,r}$.

We make the following preliminary choices and additional assumptions:

- Let $Q_3(z) = 9z^2 + 4iz^2 + \frac{8i}{3}z^3$, and $\tilde{Q}_3(z) = 4iz^2 + \frac{8i}{3}z^3$. Thus in the notation of Section 3.3.2, we have

$$M(x, y, z) = \tilde{m}(x, y) + 2Q_3(z) = \tilde{m}(x, y) + 2(3z)^2 + 2\tilde{Q}_3(z). \quad (3.3.36)$$

- For $r > 3$, let $Q_r(z)$ be a polynomial of degree r , with no constant term, such that 2 of its critical points are the same as those of $Q_3(z)$. Assume $Q_3(z) + tQ_r(z)$ is Morse for all $t \in [0, 1]$.
- Recall that for every pair (p, q) , the function $h_{p,q}$ is a good real deformation of a function of the form

$$(x^{p-2} - y^2)(x^2 - cy^{q-2}) \quad (3.3.37)$$
 for some constant c . We further assume that $h_{p,q}$ is also a polynomial, and that the coefficient of x^2y^2 is still -1 . (This is already the case for $h_{3,3}$.)
- For every pair (p, q) , we already have that $h_{p,q}$ has a real minimum to which the vanishing cycle a (one of the seven in the ‘kernel’) is associated. Assume that the corresponding critical point is $(x, y) = (0, 0)$. (Again, this is already the case for $h_{3,3}$.)
- Our arguments will use some one-parameter families of polynomials involving that $h_{p,q}$ and Q_r (the parameter lies in $[0, 1]$). One can check that we can arrange for the $h_{p,q}$ and Q_r to be such that these deformations are through Morse functions.

(Without being truly restrictive, these are all somewhat technical assumptions. They are principally designed to allow us to easily track critical points as we deform polynomials.)

We’ll start by studying $\mathcal{T}_{3,3,r}$. The function $Q_3(z) + Q_r(z)$ has $r - 1$ critical points. Label them as z_1, \dots, z_{r-1} , where $z_{r-1} = 0$, and z_{r-2} is the other critical point of Q_3 . Pick any collection of vanishing paths giving that order. The function $h_{3,3}(x, y)$ has seven critical points; choose as ordered collection of vanishing paths and labels a, \dots, g following Section 3.3.2. The function

$$h_{3,3}(x, y) + Q_3(z) + Q_r(z) \quad (3.3.38)$$

has $7(r - 1)$ critical points. From Gabrielov’s algorithm, we get a distinguished collection of vanishing paths, together with a description of the corresponding vanishing cycles as matching cycles in an auxiliary Lefschetz fibration. (The fibre of this Lefschetz fibration is the generalised Milnor fibre of $\sigma_{3,3}$.) Label the vanishing cycles as (α, j) , where $\alpha = a, b, \dots, g$, and $j = 1, \dots, r - 1$, and the corresponding critical points as $C_{(\alpha,j)}$. This is a generalization of the scenario considered in Section 3.3.2, where we had $r = 3$. Now consider the function

$$M_r(x, y, z; t; l) = h_{3,3}(x, y) + (3z + txy)^2 + \tilde{Q}_3(z) + (1 - l)Q_r(z) \quad (3.3.39)$$

where $t, l \in [0, 1]$. Fixing $l = 0$ and deforming t from 0 to 1, we get (near zero) a Morsification of $\mathcal{T}_{3,3,r}$. We would like to know which of the critical points of $M_r(x, y, z; 0; 0)$ contribute to this. We expect $p + q + r - 1 = 5 + r$ of those points in total. First, note that they must include $C_{(a,j)}$ for all $j = 1, \dots, r - 1$, as those critical points, whose x and y coordinates are zero, remain fixed as we increase t . On the other hand, suppose you start with $M_r(x, y, z; 0; 0)$, and increase l to 1. By construction, this fixes $C_{(\alpha,j)}$ for all $\alpha = a, \dots, g$, and $j = r - 2, r - 1$. Moreover, we already understand the deformation from $M_r(x, y, z; 0; 1)$ to $M_r(x, y, z; 1; 1)$ given by increasing t , as it is precisely what we studied in Section 3.3.2. In particular, the critical points that are not sent to infinity are the images of $C_{(a,r-2)}$, $C_{(a,r-1)}$, $C_{(b,r-1)}$, $C_{(c,r-1)}$, \dots , $C_{(g,r-1)}$, and the generalised Milnor fibre of $M_r(x, y, z; 1; 1)$ is $\mathcal{T}_{3,3,3}$. On the other hand, notice that deforming from $M_r(x, y, z; 1; 0)$ to $M_r(x, y, z; 1; 1)$ realizes the adjacency $\mathcal{T}_{3,3,r} \rightarrow \mathcal{T}_{3,3,3}$. Putting all of this together, we see that the Milnor fibre $\mathcal{T}_{3,3,r}$ is a subset of

the generalized Milnor fibre of $M(x, y, z; 0, 0)$, and that its critical points correspond to $C_{(a,1)}$, $C_{(a,2)}$, \dots , $C_{(a,r-2)}$, $C_{(a,r-1)}$, $C_{(b,r-1)}$, \dots , $C_{(g,r-1)}$.

We need to find vanishing paths that are compatible with the deformations. Start with $C_{(a,r-2)}$, $C_{(a,r-1)}$, $C_{(b,r-1)}$, \dots , $C_{(g,r-1)}$. From the study of $T_{3,3,3}$, we already know what vanishing paths to choose in order to avoid the exit trajectories of $C_{(b,r-2)}$, $C_{(c,r-2)}$, \dots , $C_{(g,r-2)}$. Moreover, notice that none of the cycles (α, j) , with $j = r - 2$ or $r - 1$, have intersection points any of the other cycles whose critical values exit the picture. (This can just be read off from the Gabrielov description, see Section 3.1.4.) Thus, after the mutations that we found for the $T_{3,3,3}$ case (Section 3.3.2) and maybe some extra trivial mutations, we can get vanishing paths for $C_{(a,r-2)}$, $C_{(a,r-1)}$, $C_{(b,r-1)}$, \dots , $C_{(g,r-1)}$ avoiding *all* exit trajectories. Their order will still follow that for $T_{3,3,3}$. As for $C_{(a,1)}, \dots, C_{(a,r-3)}$, one can simply start with the vanishing paths given by the Gabrielov configurations, and modify them all by the same sequence of mutations as the one that is applied to $C_{(a,r-2)}$. (There are different ways of seeing this; the simplest might be to observe that one can deform $Q_r(z)$ so that all $r - 2$ first critical points almost coincide, and consider the effect for critical points and vanishing paths for $M_r(x, y, z; t; l)$.) We're now almost done, although the description we have isn't yet as nice as the one stated in the theorem: we are at the analogue of the end of Section 3.3.2 for the $T_{3,3,3}$ case. To get the claimed configuration, we make some further mutations following Section 3.3.2 for $T_{3,3,3}$.

We are left with the case of a 'general' $T_{p,q,r}$. One can proceed similarly to before, considering the function

$$N_{p,q,r}(x, y, z; t, l, \mu) = h_{p,q}(x, y) + (1 - \mu)h_{3,3}(p, q) + (3z + txy)^2 + (1 - l)Q_r(z) \quad (3.3.40)$$

with $t, l, \mu \in [0, 1]$. We can use the Gabrielov and A'Campo techniques to describe the generalised Milnor fibre of $N_{p,q,r}(x, y, z; 0; 0; 0)$ as the total space of a Lefschetz fibration, with fibre the Milnor fibre of $\sigma_{p,q}$. For $t = 1, l = \mu = 0$, we get a Morsification of $T_{p,q,r}$. (This is where the assumption about the coefficient of x^2y^2 in the expression for $h_{p,q}$ helps, as the two terms involving x^2y^2 in $N_{p,q,r}(x, y, z; 1; 0; 0)$ cancel out.) Fixing $t = 1, \mu = 0$ and deforming l from 0 to 1 realises the adjacency $T_{p,q,r} \rightarrow T_{p,q,2}$, or the appropriate generalization in the cases where $(p, q) = (3, 4)$ or $(3, 5)$. Fixing $t = 1, l = 0$ and deforming μ from 0 to 1 realises to adjacency $T_{p,q,r} \rightarrow T_{3,3,r}$. Combining the information from both of these and proceeding similarly to above yields the desired description of vanishing cycles for $T_{p,q,r}$.

Finally, in all cases the claim about R_2 can be established completely analogously to Lemma 3.3.2. \square

Essentially local changes of the symplectic form and product structure near M_\star .

Throughout the previous section, as well as Sections 3.3.1 and 3.3.2, we considered Lefschetz fibrations $\pi : E \rightarrow B$ where E is an open subset of X , some smooth hypersurface $X \subset \mathbb{C}^3$, B is an open subset of \mathbb{C} , and π is given by a complex polynomial. The Milnor fibre of $T_{p,q,r}$ was given either by $\mathcal{T}_{p,q,r} = E$, or by an open subset $\mathcal{T}_{p,q,r}$ of E . We repeatedly displaced vanishing cycles in the fibre $\pi^{-1}(\star)$ by Hamiltonian isotopies, obtained by making essentially local changes of the symplectic form. (See Lemma 3.1.22 and thereafter, as well as the remarks in Section 3.1.2 about using these tools for more general Lefschetz fibrations.) One might worry about the effect of these changes on E as a symplectic manifold. Thankfully, we

have the following:

Lemma 3.3.4. *Let ω be the original symplectic form on $\mathcal{T}_{p,q,r}$, and ω' the form after all the essentially local changes. We claim that $(\mathcal{T}_{p,q,r}, \omega')$ is also a copy of the Milnor fibre of the same singularity, possibly defined with a different holomorphic representative of the singularity, and different cutoffs. In particular, they are exact symplectomorphic when completed with cylindrical ends.*

Proof. Changes by compactly supported one-forms are taken care of by a Moser argument. Observe that in the cases we are concerned about, $\pi : E \rightarrow B$ is always given by projecting to the third complex coordinate, z . Recall that ω , the symplectic form on E , is the restriction of usual Kaehler form on \mathbb{C}^3 :

$$\omega_{\mathbb{C}^3} = \frac{i}{4} dd^c (||x||^2 + ||y||^2 + ||z||^2). \quad (3.3.41)$$

This means that $\omega + c\pi^*\omega_b$ is given by the restriction of the Kaehler form

$$\omega_{\mathbb{C}^3} = \frac{i}{4} dd^c (||x||^2 + ||y||^2 + ||c'z||^2) \quad (3.3.42)$$

for some constant c' . This is the same effect as a holomorphic reparametrization, which does not affect completed Milnor fibres, by Lemma 3.1.5. \square

With the preceding observation under our belt, using Lemma 3.1.25, we can assume that the description of the Milnor fibre $\mathcal{T}_{p,q,r}$ given by Proposition 3.3.3 has the following additional feature:

Assumption 3.3.5. *Fix a large compact subset of M_\star , say K , such that each of the vanishing cycles on M_\star are contained in the interior of K . Fix $r > 0$ such that the closure of $B_r(\star)$ does not contain any critical values. We assume that near $K \subset M_\star$ our symplectic form restricts to ‘product’ symplectic form, say ω_{pr} , following the description of Lemma 3.1.25. Moreover, we assume that all the matching paths through \star are given by straight-line segments inside $B_{r/2}(\star)$, and that no other matching paths intersect $B_r(\star)$.*

We modify our original ω -compatible almost complex structure J following Lemmas 3.1.24 and 3.1.26. The result is an ω_{pr} -compatible almost complex structure, say J_{pr} , which on a neighbourhood of K is given by the product of a complex form on fibres and a complex form on the base.

3.4 Torus construction

3.4.1 Lagrangian surgery

We shall use the operation of Lagrangian surgery, introduced by Polterovich [55]. Suppose you have two Lagrangians L_1 and L_2 that intersect transversally at one point u . Fix an order of L_1 and L_2 , say (L_1, L_2) . Lagrangian surgery is a local procedure for obtaining a new Lagrangian $L_1 \# L_2$, which agrees with the union of L_1 and L_2 outside an arbitrarily small neighbourhood of u . Pick a Darboux neighbourhood of u , say $\phi : U \ni u \rightarrow \mathbb{R}^4$ such that

- $\phi(u) = 0$;
- $\phi(L_1 \cap U) = (\mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\}) \cap \phi(U)$;
- $\phi(L_2 \cap U) = (\{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R}) \cap \phi(U)$.

(See [55, Section 4].) Let $h : \mathbb{R} \rightarrow \mathbb{R}^2$ be a smooth embedding such that $\text{Im}(h)$ agrees with $\mathbb{R}_+ \times \{0\} \cup \{0\} \times \mathbb{R}_-$ outside $\phi(U)$, and such that there is no $x \in \mathbb{R}^2$ such that both x and $-x$ lie in $\text{Im}(h)$. The Lagrangian handle associated to h is

$$H = \{(x \cos t, y \cos t, x \sin t, y \sin t) \mid (x, y) \in \text{Im}(h), t \in S^1\}. \quad (3.4.1)$$

Now $L_1 \# L_2$ is defined by replacing the neighbourhood of u with H :

$$L_1 \# L_2 := ((L_1 \cup L_2) \setminus U) \cup \phi^{-1}(H \cap \phi(U)). \quad (3.4.2)$$

Up to Lagrangian isotopy, this only depends on our choice of ordering of L_1 and L_2 . Nevertheless, define the *parameter* of the surgery to be the area ϵ between the image of h and the union of the real and imaginary axes. (This will matter later for exactness.) Note ϵ could be negative.

3.4.2 Main construction

The vanishing cycles A and B intersect at two points, u and v . Performing Lagrangian surgeries at both u and v gives a Lagrangian torus; we shall use one such torus for our main theorem. A few preliminary remarks:

- Up to Lagrangian isotopy, there are a total of four choices.
- The topology of the result is independent of these four choices.
- It is important that the orientation of the intersections at u and v agree: otherwise, the result of the two surgeries would be a Klein bottle rather than a torus.
- We shall see that two of the four choices naturally give exact tori. However, for applications regarding some properties of the Fukaya category, we will prefer one of these choices over the other.

We want to choose two Darboux charts for each of u and v : one for the surgery with order (A, B) , and one for the surgery with order (B, A) . By Assumption 3.3.5 (product symplectic form near M_\star), we can take our Darboux charts at u to be given by the product of a Darboux chart for some open neighbourhood of u in M_\star with a Darboux chart for a neighbourhood of $\pi(u)$ in $B_r(\star)$. We order our coordinates so that open subsets of fibres correspond to subsets of $\mathbb{R}^2 \times \{(a, b)\}$, and lifts of open neighbourhoods of \star to subsets of $\{(0, 0)\} \times \mathbb{R}^2$. (In particular, π is projection to the final two coordinates.) We pick similar Darboux charts for v .

Exactness

Define f_A to be a smooth function on $A|_{M_\star}$ such that $df_A = i_{A|_{M_\star}}^* \theta$, and similarly for f_B .

Proposition 3.4.1. *Lagrangian surgeries on u and v with the same ordering of A and B , and the same parameter $\epsilon_u = \epsilon_v = \epsilon$, produce an exact torus if and only if*

$$f_A(u) - f_B(u) = f_A(v) - f_B(v). \quad (3.4.3)$$

Proof. It's equivalent to find conditions under which θ integrates to zero about two distinct primitive simple closed curves. The meridional S^1 of the torus (which vanishes when deforming back to $A \cup B$) bounds a Lagrangian disc (in the handle H of the surgery), so θ integrates to zero by Stokes' theorem. It remains to check the other direction; let S_t^1 be any such curve. Assume we choose the surgery order (A, B) . Integrating along a curve that goes from u to v on A , then v to u on B , and correcting for the surgery, we find that

$$\int_{S_t^1} \theta = f_B(u) - f_B(v) + \epsilon_u + f_A(v) - f_A(u) - \epsilon_v. \quad (3.4.4)$$

In the case where $\epsilon_u = \epsilon_v = \epsilon$, setting this expression equal to zero yields the desired result. \square

Let r_1 and r_2 be the intersection points of R_1 with, respectively, A and B . Let D_1 and D_2 be the holomorphic discs between u, r_1 and v, r_1 and v, r_2 and r_2 (shaded in Figure 3-25).

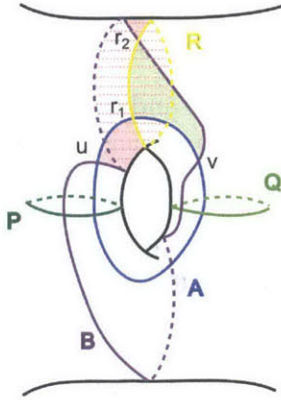


Figure 3-25: Discs D_1 and D_2 .

Using Stokes' theorem, we get the following:

Corollary 3.4.2. *Lagrangian surgery on u and v with same orderings, and same surgery parameter ϵ , produces an exact torus if and only if the discs D_1 and D_2 have the same symplectic area.*

Notice that after exact Hamiltonian isotopy of the vanishing cycles on the Riemann surface M_* , we can arrange for the two holomorphic discs D_1 and D_2 to have the same symplectic areas, while keeping minimal intersection between the curves. Thus, after an essentially local change of the symplectic form, we can arrange for this to be the case. We shall assume hereafter that these areas are equal, for any of the $T_{p,q,r}$.

Maslov class

Lemma 3.4.3. [60, Lemma 2.14] *Suppose you have two graded Lagrangians \tilde{L}_0 and \tilde{L}_1 which intersect transversally at a single point x . If we have*

$$\tilde{I}(\tilde{L}_0, \tilde{L}_1; x) = 1 \tag{3.4.5}$$

then there is a grading on $L_0 \# L_1$ which agrees with \tilde{L}_0 on $\tilde{L}_0 \cap (L_0 \# L_1)$, and with \tilde{L}_1 on $\tilde{L}_1 \cap (L_0 \# L_1)$.

As a corollary, we get:

Corollary 3.4.4. *Suppose you have two Lagrangians L_0 and L_1 with vanishing Maslov classes, and which intersect transversally at two points u and v , with agreeing orientations. Let Σ be the result of the Lagrangian surgeries at u and v , with same order (L_0, L_1) . Then Σ has vanishing Maslov class if and only if*

$$\tilde{I}(\tilde{L}_0, \tilde{L}_1; u) - \tilde{I}(\tilde{L}_0, \tilde{L}_1; v) = 0. \tag{3.4.6}$$

(The left hand side is independent of the choices of graded lifts \tilde{L}_0 and \tilde{L}_1 .)

In particular, we get the following result.

Lemma 3.4.5. *Suppose we do Lagrangian surgery at u and v with the same order for A and B . Then the resulting torus has vanishing Maslov class.*

Proof. Pick a path γ_1 from u to v along A , and γ_2 from u to v along B . We choose the ones that lie on the Riemann surface M_\star , and go through r_1 or r_2 exactly once. Pick a trivialization of the tangent space of $\pi^{-1}(B_\star)$, some small open neighbourhood of \star , given by the product of a trivialization of the tangent space of the fibre with the standard trivialization of the tangent space of the base. (This extends to a trivialization of the tangent space of the total space.) Let $\mathcal{L}(2)$ be the Grassmannian of Lagrangian planes in \mathbb{R}^4 . The path γ_i induces a path Γ_i in \mathcal{L}_2 . By Corollary 3.4.4, it's enough to show that the Maslov index of Γ_1 relative to Γ_2 vanishes. The two base components of the Lagrangians are always transverse; moreover, we can choose our trivialization such that their fibre-wise components are too (e.g., in Figure 3-25, using parallel copies of A as part of a trivialization of a neighbourhood of R). In particular, the relative Maslov index of the paths that we are considering vanishes. \square

Conclusion

Putting together the results from the previous subsections, we now see that we have:

Theorem 3.4.6. *There exists an exact Lagrangian torus T of vanishing Maslov class in the Milnor fibre of any $T_{p,q,r}$ singularity. Its homology class is given by the difference of the classes of the vanishing cycles A and B .*

As a corollary, we obtain:

Theorem 3.4.7. *The Milnor fibre of any positive modality isolated hypersurface singularity of three variables contains an exact Lagrangian torus, primitive in homology, and with vanishing Maslov class.*

Proof. This follows from Theorem 3.4.6, together with Theorem 3.1.14 (any positive modality singularity is adjacent to a parabolic singularity, which are the three simplest of the $T_{p,q,r}$), and Lemma 3.1.10 (if a singularity f is adjacent to another one, say g , there is an exact embedding of their Milnor fibres: $M_g \hookrightarrow M_f$). The homology class of the torus T in $\mathcal{T}_{p,q,r}$ is the difference of the classes of two vanishing cycles. This remains true under the embedding given by adjacency. Thus all the tori we construct have primitive homology classes. This leaves the Maslov class claim. Let f be positive modality singularity under consideration, and g a parabolic singularity that it is adjacent to. As $c_1(M_f) = 0$, there is a lift from Lagrangian Grassmanian $LGr(M_f)$ to the Grassmanian of graded Lagrangian planes, $\widetilde{LGr}(M_f)$. This restricts to a lift from $LGr(M_g)$ to $\widetilde{LGr}(M_g)$. Consider any closed path on T . This gives a path $\gamma : S^1 \rightarrow LGr(M_g)$. As T has Maslov class zero, γ lifts to a closed path $S^1 \rightarrow \widetilde{LGr}(M_g)$. (Of course, the lift of $LGr(M_g)$ to $\widetilde{LGr}(M_g)$ might be different from the one coming from trivialization of $T(M_g)$ used to calculate the Maslov class of T – but γ will lift to a closed path for *any* lift of $LGr(M_g)$ to $\widetilde{LGr}(M_g)$.) Thus the image of T in M_f also has vanishing Maslov class. \square

3.4.3 A local model for the Lagrangian surgeries

For some Floer–theoretic computations, it will later be useful to have the following local model for the Lagrangian surgeries we make. Consider the Lefschetz fibration

$$\chi : \tilde{\mathcal{C}} := \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^2 + z^2 = 1\} \rightarrow \mathbb{C} \quad (3.4.7)$$

$$(x, y, z) \mapsto z. \quad (3.4.8)$$

The smooth fibre is a cone. There are two critical values, $z = \pm 1$, and the fibre above each of them is the union of two lines. Define

$$\mathcal{C} := \tilde{\mathcal{C}} \setminus \chi^{-1}(0) \quad (3.4.9)$$

equipped with the exact symplectic form associated to the plurisubharmonic function

$$h(x, y, z) = |x|^2 + |y|^2 + (\log|z|)^2. \quad (3.4.10)$$

In particular, we have

$$\omega = \frac{i}{2} \left(dx \wedge d\bar{x} + dy \wedge d\bar{y} + \frac{dz \wedge d\bar{z}}{z\bar{z}} \right). \quad (3.4.11)$$

(The reader might want to think of the z coordinate as an infinite annulus. Setting $z = e^{s+it}$, the z -terms of ω are a multiple of $ds \wedge dt$.) Also, the z -terms of θ (say θ_z) are a multiple of $\log|z|$. In particular,

$$\int_{|z|=1} \theta_z = 0. \quad (3.4.12)$$

The two unit half-circles in \mathbb{C}^* give matching paths between $z = 1$ and $z = -1$. (See Figure 3-26.) Using symplectic parallel transport, to each of these corresponds a Lagrangian sphere in \mathcal{C} . (See e.g. [42].) By for instance symmetry considerations, we do not even need to modify the symplectic form to get these. Call them \mathcal{A} and \mathcal{B} . A Darboux-type argument gives:

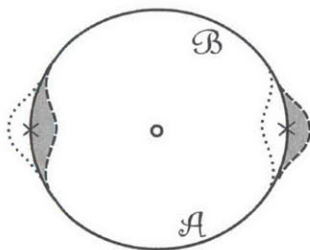


Figure 3-26: Local model for Lagrangian surgeries on the intersections of A and B .

Proposition 3.4.8. *There is an exact symplectomorphism from an open neighbourhood of the union of A and B to an open neighbourhood of the union of \mathcal{A} and \mathcal{B} , mapping A to \mathcal{A} and B to \mathcal{B} .*

(Exactness follows from Equation 3.4.12, which ensures both intersection points have equal action.) Near 1 or -1 , replace the union of the two matching paths by a curve segment avoiding the singular value (in each case, either to its left or its right). These are the dotted and dashed segments on our figure. The result, an S^1 in the base, gives a Lagrangian torus in the total space by taking the symplectic parallel transport of the vanishing cycle around S^1 . (Here you might have to make an essentially local change to the symplectic form to get ends to match up.) Up to Lagrangian isotopy, there are four choices. If you make the perturbations towards to same side (either left or right – both dashes or both dots), the result will be exact if and only if the two displaced areas in the base agree. (These are the shaded regions in our figure. Their areas are calculated with respect to the symplectic form given by the z -terms of ω .) One can check that:

Proposition 3.4.9. *The four Lagrangian tori obtained by surgery on u and v corresponds to the four matching tori described above. Moreover, the displaced area agrees with the surgery parameter.*

3.4.4 Tori in parabolic singularities

Main result and discussion

Proposition 3.4.10. [30] *The three parabolic singularities have an semi-definite intersection form, with a two-dimensional nullspace.*

So far, we've been considering the null-class given by $[5] - [4]$ in the notation of the Dynkin diagram of Figure 3-2 – that is, the class $[A] - [B]$ in the notation of e.g. Section 3.3. There is a second independent class with a nice description in terms of the Dynkin diagram 3-2: after quotienting out the class $[A] - [B]$, it is given by a weighted combination of vertices of the quotient diagram. See Figure 3-27 for the case of $T_{6,3,2}$.

Was the null-class that we constructed our torus in special in any way?

Theorem 3.4.11. *For parabolic singularities, there is an exact Lagrangian torus with vanishing Maslov class in each primitive homology classes in the null-space of the intersection form.*

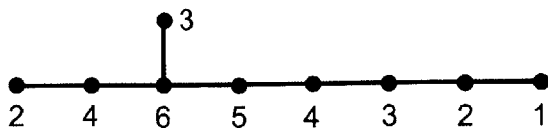


Figure 3-27: Second null-class for the intersection form of $T_{6,3,2}$. The integers denote weights.

This means that there are tori in all the topologically permissible primitive homology classes for parabolic singularities. The case of non-primitive classes remains open.

The proof presented here uses compactifications of the three relevant Milnor fibres to del Pezzo surfaces, which might also prove of independent interest.

Points in almost general position and del Pezzo surfaces

Let us start with some preliminaries.

Definition 3.4.12. *A del Pezzo surface is a smooth projective surface with an ample anti-canonical bundle. The rank of a del Pezzo is the self-intersection of its anti-canonical class.*

To check that a surface is del Pezzo, it is enough to calculate the self-intersection of an anti-canonical divisor D , and to check that for any irreducible curve C , $D \cdot C > 0$ (Nakai-Moishezon criterion). Moreover, there is a short classification of these surfaces by rank, using the following notion:

Definition 3.4.13. *A collection of points Γ in \mathbb{P}^2 are in almost general position if no three points lie on a line; no six points lie on a conic; and there does not exist a cubic curve passing through seven of the points and with a double point at the eighth. (See e.g. [20, Exposé 2].)*

Note that there can be at most eight points in general position in \mathbb{P}^2 . We shall use:

Theorem 3.4.14. *(see e.g. the survey [20]) Any rank one del Pezzo surface is the blow-up of \mathbb{P}^2 at eight points in almost general position; rank two, seven points; and rank three, six points.*

Additionally, we shall make use of the following results about points in general position.

Lemma 3.4.15. [20] *Consider any collection of points in \mathbb{P}^2 in almost general position. Then there is a smooth cubic curve that contains all of them.*

Conversely, fix any smooth cubic curve E in \mathbb{P}^2 , and an integer n , $1 \leq n \leq 8$. Consider the collection of all sets of n distinct (unordered) points on E . This is an algebraic variety: $(E^n - V)/Sym^n$, where V is a closed subvariety of E^n .

Lemma 3.4.16. *The collection of sets of n points on E in almost general position, say E_{gp}^n , is an open, connected, non-empty subvariety of $(E^n - V)/Sym^n$.*

Proof. Points satisfying each of the conditions in Definition 3.4.13 give proper closed subvarieties of E^n , invariant under the action of Sym^n . \square

We shall actually need a version of this for a one-parameter family of elliptic curves.

Definition 3.4.17. Let \mathcal{E} be the collection of all smooth cubic curves in \mathbb{P}^2 (that is, the space of coefficients, with the discriminant, corresponding to singular curves, removed).

As a smooth complex variety, \mathcal{E} has a natural smooth structure. Let $\lambda : S^1 \rightarrow \mathcal{E}$ be any smooth loop (we use the parametrization $S^1 = \mathbb{R}/\mathbb{Z}$). Suppose that you have a collection Γ_0 of n points in almost general position on $\lambda(0)$.

Lemma 3.4.18. We can find a smooth map

$$\Gamma : \bigsqcup_n [0, 1] \rightarrow \mathbb{P}^2 \quad (3.4.13)$$

such that $\Gamma(\{0\}, \dots, \{0\}) = \Gamma(\{1\}, \dots, \{1\}) = \Gamma_0$ (set-wise), and $\Gamma(\{t\}, \dots, \{t\})$ is a collection of points in general position in the cubic curve $\lambda(t)$ for every t .

Proof. Starting with Γ_0 , an extension to a path of n points certainly exists. Also, for any cubic E , $(E^n - V)/\text{Sym}^n$ is a smooth complex manifold, and E_{gp}^n a (Zariski) open subvariety. Thus, given any path of cubic curves, the space of paths of n points in almost general position is a (topologically) dense open subset of the space of paths of n points. Moreover, in the case of a loop of cubic curves, as E_{gp}^n is connected, we can arrange for the set of n points at $t = 1$ to match the initial one. \square

Compactification to del Pezzo surfaces

Each of the parabolic singularities has an isolated singularity at the origin, and no other singular values. Thus the Milnor fibre of $x^3 + y^3 + z^3$ is represented by the hypersurface

$$x^3 + y^3 + z^3 + 1 = 0 \quad (3.4.14)$$

and similarly for the other two.

Proposition 3.4.19. We have that:

- $x^3 + y^3 + z^3 + 1 = 0$ compactifies to a rank three del Pezzo surface inside \mathbb{P}^3 .
- $x^4 + y^4 + z^2 + 1 = 0$ compactifies to a rank two del Pezzo surface inside the weighted projective space $\mathbb{P}^3(1, 1, 2, 1)$.
- $x^6 + y^3 + z^2 + 1 = 0$ compactifies to a rank one del Pezzo surface inside the weighted projective space $\mathbb{P}^3(1, 2, 3, 1)$.

In each case, the anti-canonical divisor given by intersecting with the hyperplane at infinity is a smooth elliptic curve.

Proof. Case of (3,3,3). Compactify $x^3 + y^3 + z^3 + 1 = 0$ inside \mathbb{P}^3 to $x^3 + y^3 + z^3 + w^3 = 0$, say \overline{P}_8 . The divisor at infinity (i.e. the intersection with $w = 0$), say D , is an elliptic curve: a cubic inside \mathbb{P}^2 . It is an anti-canonical divisor: the form defined by

$$\Omega \wedge df = dx \wedge dy \wedge dz \quad (3.4.15)$$

where $f(x, y, z) = x^3 + y^3 + z^3$, trivialises the canonical bundle of its complement. It is then immediate to check that $D \cdot D = 3$, and that $D \cdot C > 0$ for any irreducible surface C in \overline{P}_8 .

Case of (4,4,2). Compactify $x^4 + y^4 + z^2 + 1 = 0$ inside the weighted projective space $\mathbb{P}^3(1, 1, 2, 1)$ to $x^4 + y^4 + z^2 + w^4 = 1$, say \overline{X}_9 . This avoids the singular points of the weighted projective space. Again, the intersection with $w = 0$ is an anticanonical divisor, say D . It's also an elliptic curve; one can for instance use the fact that it is branch-double-covered by the quartic $x^4 + y^4 + z^4$ in \mathbb{P}^2 . We claim the total space is a del Pezzo surface of rank 2. Here is an elementary way of computing the self-intersection of D : consider the map $\pi : \mathbb{P}^3(1, 1, 2, 1) \rightarrow \mathbb{P}^2$ given by $[x : y : z : w] \mapsto [x : y : w]$. The surface \overline{X}_9 double covers \mathbb{P}^2 , with branching over a quartic curve Q . By the generalised Riemann–Hurwitz theorem,

$$K_{\overline{X}_9} = \pi^* K_{\mathbb{P}^2} + R \quad (3.4.16)$$

where R is the ramification divisor upstairs. Notice that R has the same support as π^*Q , but different multiplicity: $\pi^*Q = 2R$. Thus

$$2K_{\overline{X}_9} = \pi^*(2K_{\mathbb{P}^2} + Q) = \pi^*\mathcal{O}(-2) \quad (3.4.17)$$

Thus $-K_{\overline{X}_9}$ is ample, and

$$D \cdot D = \frac{1}{4} \deg(\pi) \times 4 = 2. \quad (3.4.18)$$

Case of (6,3,2). Similarly, compactify $x^6 + y^3 + z^2 + 1 = 0$ inside the weighted projective space $\mathbb{P}^3(1, 2, 3, 1)$ to $x^6 + y^3 + z^2 + w^6 = 1$, say \overline{J}_{10} . The anti-canonical divisor given by the intersection with $w = 0$, say D , is still an elliptic curve (for instance, one can consider its branch-covering by the sextic $x^6 + y^6 + z^6$ in \mathbb{P}^2). The total space is a del Pezzo surface of rank 1. Why? You can use a different auxiliary projection to \mathbb{P}^2 : the map $\pi : \mathbb{P}^3(1, 2, 3, 1) \rightarrow \mathbb{P}^2$ given by $[x : y : z : w] \mapsto [x : y^2 : w]$. The surface \overline{J}_{10} double covers \mathbb{P}^2 , with branching over two copies of \mathbb{P}^1 . Proceeding similarly to the case of \overline{X}_9 , we find that $K_{\overline{J}_{10}} = \pi^*\mathcal{O}(-1)$. \square

Variation operator and the nullspace of the intersection form

Our next ingredient is a characterization on the nullspace of the intersection form. Let S be the compactification of a parabolic Milnor fibre to a del Pezzo surface, D the preferred anti-canonical divisor in S , and $M^c := S \setminus \nu(D)$ the complement of a neighbourhood of D (that is, the compact version of the Milnor fibre $M = S \setminus D$).

Proposition 3.4.20. *Let $i : H_2(\partial M^c) \rightarrow H_2(M)$ be the map induced by inclusion. It is injective, and that the classes in the nullspace of the intersection form are precisely its image.*

Proof. It is equivalent to prove this for $i : H_2(\partial M^c) \rightarrow H_2(M^c)$. Injectivity follows from the long exact sequence of the pair $(M^c, \partial M^c)$, together with the fact that $H_3(M^c, \partial M^c) \cong H^1(M^c) = 0$. For $\alpha \in H_2(M^c)$, the form $\langle \cdot, \alpha \rangle \in \text{Hom}(H_2(M^c), \mathbb{Z})$ is given by the image of α under the standard maps

$$H_2(M^c) \rightarrow H_2(M^c, \partial M^c) \cong (H_2(M^c))^*. \quad (3.4.19)$$

Using the long exact sequence of the pair $(M^c, \partial M^c)$ again, we see that $\langle \cdot, i(\beta) \rangle$ vanishes for any $\beta \in H_2(\partial M^c)$. \square

Monodromy of loops of smooth cubics curves

Here is the final ingredient.

Lemma 3.4.21. *Fix a loop $\lambda : S^1 \rightarrow \mathcal{E}$, and let $E = \lambda(0)$. The isomorphism $H_1(E) \rightarrow H_1(E)$ that this induces can correspond to any element $\gamma \in SL_2(\mathbb{Z})$.*

Proof. This follows from the usual action of $SL_2(\mathbb{Z})$ on the upper-half plane. As the space of cubic curves is path-connected, it is enough to understand this for any fixed cubic curve. Pick a point τ in the upper-half plane. It determines an (abstract) elliptic curve E . Three distinct ordered marked points (say p, q and r) in the fundamental domain associated to τ specify an embedding into \mathbb{P}^2 : they determine a degree three line bundle on E with a preferred ordered basis of global sections. Remember γ is any element of $SL_2(\mathbb{Z})$. We would get the same cubic in \mathbb{P}^2 by using $\gamma(\tau)$ and marked points $\gamma(p), \gamma(q)$ and $\gamma(r)$. Now pick a path $\tau(t)$ in the upper-plane between $\tau(0) = \tau$ and $\tau(1) = \gamma(\tau)$, and paths of points $p(t), q(t)$ and $r(t)$ in the fundamental domain of $\tau(t)$, with similar conditions. This determines a path of cubic curves in \mathbb{P}^2 ; by construction, it has the required property. \square

Remark 3.4.22. *There is more to the homotopy group $\pi_1(\mathcal{E})$ than $SL_2(\mathbb{Z})$: for instance, there is a semi-direct product with $\mathbb{Z}/3 \times \mathbb{Z}/3$ (think about which triples of points determine the same embedding), which doesn't get detected by the action on H_1 . For a full description of $\pi_1(E)$, see [48].*

Conclusion of argument

As before, let S be a del Pezzo surface that is the compactification of a parabolic Milnor fibre, and let D be the preferred anti-canonical divisor in S . Blow down six, seven or eight (say n) exceptional curves on S to get to \mathbb{P}^2 . The exceptional curves blow down to a collection of n points in almost general position in \mathbb{P}^2 , say Γ . Moreover, D is the proper transform on a cubic curve E in \mathbb{P}^2 passing through all points of Γ . Now pick an element $\gamma \in SL_2(\mathbb{Z})$. By Lemma 3.4.21, we can find a loop $\lambda : S^1 \rightarrow \mathcal{E}$ with $\lambda(0) = E$ such that the resulting isomorphism $H_1(E) \rightarrow H_1(E)$ is given by γ . Moreover, by Lemma 3.4.18, we can smoothly extend Γ to a one-parameter family of n points in almost general position $\Gamma(t) \subset \lambda(t)$.

Now consider $S^1 \times \mathbb{P}^2$. In $\{t\} \times \mathbb{P}^2$, blow up $\Gamma(t)$. The result is a smooth S^1 -bundle with fibre a del Pezzo surface of fixed rank. Moreover, each fibre, say $S(t)$, comes with a complex structure and favourite anti-canonical divisor, say $D(t)$, which is the proper transform of the cubic $\lambda(t)$. Notice that $S(0) = S(1) = S$, and similarly with D . Call this fibre bundle \mathcal{S} .

Choosing smooth families of global sections, we can construct an embedding $\mathcal{S} \subset S^1 \times \mathbb{P}^k$ such that $S(t) \subset \{t\} \times \mathbb{P}^k$ is a projective embedding determined by the square of the anti-canonical bundle. Moreover, composing with a loop in $PGL_{k+1}(\mathbb{C})$, we can assume that the image of $D(t)$ is always the intersection of $Im(S(t))$ with the hyperplane $x_{k+1} = 0$. Removing these, we get a smooth S^1 -bundle \mathcal{M} with fibres diffeomorphic to M , the Milnor fibre we started with. Moreover, each fibre $M(t)$ comes equipped with an embedding to \mathbb{C}^k ; in particular, it comes with a favourite exact symplectic form, inherited from the Kaehler form on \mathbb{C}^k . By construction, $M(0) = M(1)$ is just the Milnor fibre M we started with (as an exact symplectic manifold).

Claim 3.4.23. *The monodromy map $f : M \rightarrow M$ of \mathcal{M} acts on the nullspace of the intersection form by our chosen element $\gamma \in SL_2(\mathbb{Z})$.*

Proof. By construction, we know that γ gives a map $H_1(E) \rightarrow H_1(E)$; within the fibre bundle \mathcal{S} , one can consider the smooth automorphism of D obtained by following the path of favourite anti-canonical divisors $D(t)$. By construction, the action on first homology is also given by γ . Now consider the compact version of the Milnor fibre $M^c := S \setminus \nu(D)$. (This is the same notation as Proposition 3.4.20.) There is a natural isomorphism $H_1(D) \cong H_2(\partial M^c)$, e.g. obtained by considering the Gysin sequence. In particular, the monodromy action on $H_1(D)$ is the same as the monodromy action on $H_2(\partial M^c)$. By Proposition 3.4.20, we are done. \square

The final step is to show that we can use this monodromy to take the exact Lagrangian torus that we already have to another one. Of course, the monodromy is only a smooth map; we need to proceed with a little caution.

Claim 3.4.24. *Start with the exact Lagrangian torus $T \subset M$ of Theorem 3.4.7. We claim we can find an exact Lagrangian torus T' that is isotopic to $f(T)$, and has vanishing Maslov class.*

Proof. Let $\omega(t)$ be the Kaehler symplectic form on $M(t)$, and set $\omega = \omega(0) = \omega(1)$ on M . Pick a smooth family

$$\psi : [0, 1] \times M \rightarrow \mathcal{M} \tag{3.4.20}$$

such that $\psi(t)$ is a diffeomorphism $M \rightarrow M(t)$, with $\psi(0) = \text{Id}$ and $\psi(1) = f$.

Now $\omega_t := (\psi(t))^* \omega(t)$ is a smooth one-parameter family of symplectic forms on M ($t \in [0, 1]$) such that $\omega_0 = \omega(0)$ and $\omega_1 = f^*(\omega)$. We can associate to this a Moser vector-field $X(t)$ on M , whose flow, *when defined*, will preserve exact Lagrangians.

On each fibre $M(t)$, the Liouville vector field $Z(t)$ of $\omega(t)$ is just the restriction of the gradient vector field of $\sum |x_n|^2$ (the square of the distance function). In particular, for any sufficiently large R , $M(t)$ is transverse to the sphere $S(R)$, and $Z(t)$ points outwards on the boundary of $M(t)^R := M(t) \cap B_0(R)$ for all $t \in [0, 1]$.

Pick an increasing smooth function $\sigma : [0, 1] \rightarrow \mathbb{R}_+$ such that $\sigma(0) = 0$ and for any $t_1 < t_2$,

$$(\psi(t_1))^{-1}(M(t_1)^{\sigma(t_1)+R}) \subset (\psi(t_2))^{-1}(M(t_2)^{\sigma(t_2)+R}). \tag{3.4.21}$$

We claim that one can find sufficiently large constants R and c such that $T \subset \text{Int}(M^R)$, and the vector field

$$X(t) - (\psi^{-1})_*(cZ(t)) \tag{3.4.22}$$

points inwards along the boundary of $M(t)^r$ for all r with $R \leq r \leq R + \sigma(1)$, and all t . (Why? Such constants certainly exist for any t ; now use uniform continuity.)

Now consider the *time-dependent* vector field $X(t) - (\psi^{-1})_*(cZ(t))$. By construction, its flow is defined for any starting point in the interior of M^R . Let T' be the image of T ; it is an exact Lagrangian with respect to ω_1 (with the pull-back primitive one-form). Now $f(T')$ is an exact Lagrangian with respect to the original symplectic structure, and isotopic to $f(T)$. Finally, T' has vanishing Maslov class (as do the images of T at each time t), and so $f(T')$ has too. \square

This concludes the proof of Theorem 3.4.11.

3.5 The Fukaya category of unimodal Milnor fibres

3.5.1 Statements of results

Preferred torus

For the remainder of this section, we focus on the torus T obtained by making both surgeries at u and v with the order (A, B) , with equal parameters ϵ . The restriction of T to the fibre M_\star is given in Figure 3-28.

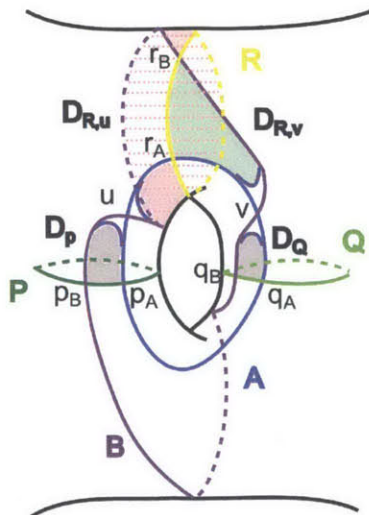


Figure 3-28: Intersection of T and M_\star : union of the A and B curves, with surgeries at u and v . The intersection points with P are p_A and p_B , similarly for Q and R . The shaded and dotted holomorphic discs are labelled D_P , D_Q , $D_{R,u}$ and $D_{R,v}$.

In particular, there are visible holomorphic discs between T and $P = P_1$, $Q = Q_1$ and $R = R_1$. We label these as in Figure 3-28. They shall turn out to be the only discs. Recall that to identify the space of equivalence classes of pairs (spin structure, flat complex line bundles) on T with $(\mathbb{C}^*)^2$, it is enough to pick a basis of $H_1(T)$. We do so as follows:

- T has a preferred meridional direction (which vanishes when deforming to $A \cup B$). We take this as the first coordinate.
- For the second coordinate, we need to choose a longitudinal S^1 . We choose the S^1 component of $T|_{M_\star}$ that is parallel to $R|_{M_\star}$ (top component in Figure 3-28).

Floer cohomologies

Proposition 3.5.1. *Pick any lift T to the Lagrangian Grassmanian of $\mathcal{T}_{p,q,r}$, say \tilde{T} ; similarly for A , B and R_1 . The Floer cohomology between the associated branes is as follows:*

$$HF^*((\tilde{T}, (\alpha, \beta)), \tilde{A} \text{ or } \tilde{B}) = \begin{cases} 0 & \alpha \neq 1 \\ H^{*+k}(S^1) & \alpha = 1 \end{cases} \quad (3.5.1)$$

$$HF^*((\tilde{T}, (\alpha, \beta)), \tilde{R}_1) = \begin{cases} 0 & \beta \neq 1 \\ H^{*+k}(S^1) & \beta = 1 \end{cases} \quad (3.5.2)$$

where the grading shift k depends on the choices of absolute gradings. (For \mathbb{Z}_2 -gradings, forget about the choices of lifts, and take $k = 0$.) Floer cohomology groups with all other vanishing cycles are identically zero.

Remark 3.5.2. In the case where T is decorated with a trivial flat complex line bundle, one could use the framework of Biran and Cornea [15] to show that $HF^*((\tilde{T}, (\pm 1, \pm 1)), -)$ lies in a long exact sequence with $HF^*(\tilde{A}, -)$ and $HF^*(\tilde{B}, -)$.

(Split-)generation of the Fukaya category

Theorem 3.5.3. The Fukaya category of the Milnor fibre of $T_{p,q,r}$ is not split-generated by any collection of vanishing cycles.

Proof. This is a consequence of Proposition 3.5.1 and Lemma 3.2.2, together with the fact that

$$HF^*((\tilde{T}, (\alpha, \beta)), (\tilde{T}, (\alpha, \beta))) \cong HF^*((\tilde{T}, (1, 1)), (\tilde{T}, (1, 1))) \cong H^*(T^2) \neq 0 \quad (3.5.3)$$

where the first isomorphism follows from definitions, and the second one from a Puinikhin–Salamon–Schwartz type argument; see e.g. work of Albers [6]. The \mathbb{Z}_2 -graded case is similar. \square

In particular, Seidel’s result (Theorem 3.2.3) is strict.

Remark 3.5.4. There are versions of Theorem 3.5.3 for the following ‘flavours’ of Fukaya categories:

- \mathbb{Z}_2 -graded, where objects are pairs (L, \mathfrak{s}_L) , with L compact orientable exact Lagrangian and \mathfrak{s}_L a spin structure on it.
- absolutely \mathbb{Z} -graded, where objects are pairs $(\tilde{L}, \mathfrak{s}_L)$, with L compact orientable exact Lagrangian of Maslov class zero, \tilde{L} a lift of L to the Lagrangian Grassmanian and \mathfrak{s}_L a spin structure on L .

With our notation, in the case of T , this is the same as restricting ourselves to the cases where $\alpha = \pm 1$, $\beta = \pm 1$.

3.5.2 Floer cohomology of T with the vanishing cycles P_1 , Q_1 and R_1

Here we prove Proposition 3.5.1 when the vanishing cycle involved is one of P_i , Q_j or R_k . The result is immediate unless i , j or k is one.

Recall that on an open neighbourhood U of K , some large compact subset in M_\star , we have a product symplectic form ω_{pr} and compactible almost complex structure J_{pr} . (The set U is contained in $\pi^{-1}(B_{r/2}(\star))$. See Assumption 3.3.5.) The Darboux charts about u and v in which we perform Lagrangian surgeries are both contained in U . With respect to these choices, there are four visible holomorphic discs between T and these vanishing cycles: one with P_1 , one with Q_1 , and two with R_1 . See Figure 3-28. To get the claimed Floer cohomology computations, we will show that with respect to this almost complex structure:

- these are the only discs between T and P_1 , Q_1 or R_1 ;
- the discs all have Maslov index one;
- the discs are regular.

We will address these points one by one. Why is this enough? It is certainly the case for P_1 and Q_1 , where there is only one disc. In the case of R_1 , the two discs contribute

$$(1 - \beta)r_1 \tag{3.5.4}$$

to the differential $\partial r_2 \in CF(R_1, (T, (\alpha, \beta)))$.

Uniqueness of discs

Give $\mathbb{R} \times [0, 1] \subset \mathbb{C}$ the usual complex structure i , and consider a holomorphic map $\sigma : \mathbb{R} \times [0, 1] \rightarrow \mathcal{T}_{p,q,r}$, such that

- $\mathbb{R} \times \{0\}$ maps to P_1 and $\mathbb{R} \times \{1\}$ maps to T (Lagrangian boundary conditions)
- $\lim_{s \rightarrow -\infty} \sigma(s, \cdot) = p_1$ and $\lim_{s \rightarrow -\infty} \sigma(s, \cdot) = p_2$ (asymptotic conditions).

These are precisely the maps used to count the coefficient of p_1 in $\partial p_2 \in CF(P_1, (T, (\alpha, \beta)))$. The disc D_P gives one such map (or, rather, a one parameter family of maps), say τ . We would like to argue that there are no other maps.

Suppose we have such a map σ . First, using the open mapping theorem, we see that $\text{Im}(\sigma)$ must lie in $\pi^{-1}(B_{r/2}(\star))$. Also, as they only depend on p_1 and p_2 , the symplectic areas for σ and τ agree:

$$\int_{\mathbb{R} \times [0, 1]} \sigma^* \omega_{pr} = \int_{\mathbb{R} \times [0, 1]} \tau^* \omega_{pr}. \tag{3.5.5}$$

Let $\pi_2 : U \rightarrow M_\star$ be the projection to the fibre above \star . By construction, this is (J_{pr}, J_\star) -holomorphic, where J_\star is the complex structure on M_\star . Also, π_2 projects A onto $A|_{M_\star}$, and similarly for B and P_1 . Now notice that

$$\int_{\mathbb{R} \times [0, 1]} \sigma^* \omega_{pr} \geq \int_{\mathbb{R} \times [0, 1]} (\pi_2 \circ \sigma)^* \omega_{M_\star}. \tag{3.5.6}$$

Moreover, $\pi_2 \circ \sigma : \mathbb{R} \times [0, 1] \rightarrow M_\star$ is (i, J_\star) -holomorphic. By topological considerations, we see that $\text{Im}(\pi_2 \circ \sigma)$ must *contain* D_P . More precisely, it must agree with D_P everywhere apart from the neighbourhood $U \subset M_\star$ of u where the surgery is performed. In U , the boundary of $\text{Im}(\pi_2 \circ \sigma)$ must be some curve in the image of the handle H under the projection π_2 . In particular, the curve belonging to T , which already lies in M_\star , is the option that gives the smallest area. Moreover, it can only be realised as a portion of the boundary of $\text{Im}(\pi_2 \circ \sigma)$ if the corresponding boundary portion of $\text{Im}(\sigma)$ already lies in M_\star . Thus the previous equation can be strengthened to:

$$\int_{\mathbb{R} \times [0, 1]} \sigma^* \omega_{pr} \geq \int_{\mathbb{R} \times [0, 1]} (\pi_2 \circ \sigma)^* \omega_{M_\star} \geq \int_{\mathbb{R} \times [0, 1]} \tau^* \omega_{pr} \tag{3.5.7}$$

with two equalities if and only if σ and τ are the same map up to translation of the real coordinate on $\mathbb{R} \times [0, 1]$. Equation 3.5.5 implies that this must be the case.

The case of D_Q is completely analogous. For $D_{R,u}$ and $D_{R,v}$, one proceeds similarly to show that the discs must lie in M_* , though there are then two possibilities. (Both of them have the same symplectic area: we meet again our assumptions on the symplectic area of the discs D_1 and D_2 , and the equality of the surgery parameters, which allowed exactness of T in the first place.)

Maslov index of discs

Pick a trivialisation of the tangent space of $\mathcal{T}_{p,q,r}$ that on $\pi^{-1}(B_{r/2}(\star))$, is given by the product of a trivialization of the tangent space of M_* with the standard trivialization of the base. Let us start by calculating the Maslov index of D_P .

Let $\gamma_1 : [0, 1] \rightarrow \mathcal{T}_{p,q,r}$ be the path from p_1 to p_2 on P along the boundary of D_P , and $\gamma_2 : [0, 1] \rightarrow \mathcal{T}_{p,q,r}$ be the path from p_1 to p_2 on T along the boundary of D_P . These determine paths $\Gamma_i : [0, 1] \rightarrow \mathcal{L}(2)$, where $\mathcal{L}(2)$ is the Lagrangian Grassmannian. The fibre-wise components of these paths contribute $+1$ to the Maslov index of Γ_2 relative to Γ_1 . It remains to show that the base components of the paths do not contribute anything. Recall the Lagrangian handle H (Section 3.4.1) is given by

$$H = \{(x \cos t, y \cos t, x \sin t, y \sin t) \mid (x, y) \in \text{Im}(h), t \in S^1\} \subset \mathbb{R}^4. \quad (3.5.8)$$

Under the identifications described in that section, γ_2 corresponds to the path

$$\{(x, y, 0, 0) \mid (x, y) \in \text{Im}(h), t \in S^1\}. \quad (3.5.9)$$

Tangent vectors to H along γ_2 are given by $(t_1, t_2, 0, 0)$, the extension of the tangent vector to $\text{Im}(h)$, and $(0, 0, x, y)$. Thus the corresponding path in the Lagrangian Grassmannian is described by Figure 3-29. It contributes zero to the relative Maslov index.

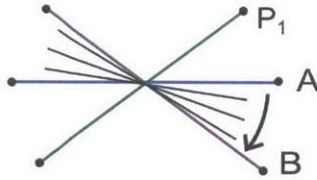


Figure 3-29: Path of Lagrangian lines determined by γ_2 , and Lagrangian line for P_1 .

The other three discs are completely analogous.

Regularity of discs

Let M denote $\mathcal{T}_{p,q,r}$, L_0 denote P_1 (or, depending on the disc we are considering, Q_1 or R_1), and L_1 the torus T . Let $R = \mathbb{R} \times [0, 1]$, and fix σ as above. The corresponding linearised

Cauchy–Riemann operator D_σ is a Fredholm operator:

$$\begin{aligned} D_\sigma &: W^{k,p}(R; \sigma^*(TM), \sigma^*(TL_0), \sigma^*(TL_1)) \\ &\rightarrow W^{k-1,p}(R; T^*R \otimes \sigma^*(TM), T^*R \otimes \sigma^*(TL_0), T^*R \otimes \sigma^*(TL_1)) \end{aligned} \quad (3.5.10)$$

for some k and p . We want to show that σ is regular, that is, that the operator D_σ is onto. From our Maslov index calculation, we already know that this operator has index one. Thus it's enough to show that the kernel of D_σ is one-dimensional. This means it must correspond to translations in the \mathbb{R} -direction of the domain of R , and nothing else. Recall D_σ is defined by extending an operator

$$\begin{aligned} D_\sigma &: C^\infty(R; \sigma^*(TM), \sigma^*(TL_0), \sigma^*(TL_1)) \\ &\rightarrow C^\infty(R; T^*R \otimes \sigma^*(TM), T^*R \otimes \sigma^*(TL_0), T^*R \otimes \sigma^*(TL_1)). \end{aligned} \quad (3.5.11)$$

The kernel of the operator on the smooth spaces is the same as the kernel on the completion. (Why? This follows from boundedness in the injective case. In general, notice that the kernel in the C^∞ space, which is finite dimensional, is already closed in the Sobolev norm. The question then reduces to the injective case.) Consider a smooth one parameter family of (j, J_{pr}) -holomorphic maps:

$$\sigma_t : \mathbb{R} \times [0, 1] \rightarrow M = \mathcal{T}_{p,q,r} \quad (3.5.12)$$

such that $t \in (-1, 1)$, $\sigma_0 = \sigma$, and each σ_t has the same boundary and asymptotic conditions as σ . Let X be the vector field such that

$$\sigma_t = \sigma_0 + tX + \text{higher order terms in } t. \quad (3.5.13)$$

It is an element of $C^\infty(R; \sigma^*(TM), \sigma^*(TL_0), \sigma^*(TL_1))$. By construction, it belongs to the kernel of D_σ . Moreover, any smooth element of the kernel arises in this way. The uniqueness argument of Section 3.5.2 show that the map

$$\pi \circ \sigma_t : R \times (-1, 1) \rightarrow \mathbb{C} \quad (3.5.14)$$

is the constant map to \star . Thus

$$0 = \frac{d}{dt}(\pi \circ \sigma_t) = D\pi\left(\frac{d}{dt}\sigma_t\right) = D\pi(X). \quad (3.5.15)$$

This means the horizontal component of X always vanishes. It remains to understand its vertical component. For this, we just use so called ‘automatic regularity’ for discs on Riemann surfaces (see e.g. [62, Section 13a]).

3.5.3 Floer cohomology of T with the vanishing cycles A and B

To complete the proof of Proposition 3.5.1, it remains to consider the case where the vanishing cycle involved is A or B . We will use the local model for our construction described in section

3.4.3. Recall we use the Lefschetz fibration:

$$\chi : \tilde{\mathcal{C}} := \{(x, y, z) \in \mathbb{C}^3 \mid x^2 + y^2 + z^2 = 1\} \rightarrow \mathbb{C} \quad (3.5.16)$$

$$(x, y, z) \mapsto z. \quad (3.5.17)$$

with $\chi^{-1}(0)$ removed. (The resulting space was called \mathcal{C} .) The cases of A and B are analogous; let us compute Floer cohomology between T and B .

After Hamiltonian isotopy, we can arrange for the curve defining T , and the matching path for B , to intersect in exactly one point in \mathbb{C}^* , say w , as in Figure 3-30.

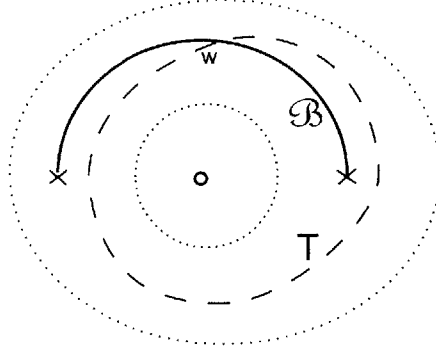


Figure 3-30: Local model for the intersection of T and B : base.

After a further Hamiltonian isotopy (with an essentially local change of the symplectic form, for instance), we can assume that on $\chi^{-1}(w)$, which is a copy of T^*S^1 , T and B restrict to two meridional S^1 's that intersect in two points, and are Hamiltonian deformations of each other. There are two 'immediate' holomorphic discs between the two intersection points, say b_1 and b_2 , corresponding to the two closed regions bounded by the union of the two S^1 's. As before, we want to prove that these are the only holomorphic discs.

Consider the standard complex structure J on \mathcal{C} . Choose a tubular neighbourhood of the Lagrangian spheres $\mathcal{A} \cup \mathcal{B}$, say ν , such that $\partial\nu$ is of contact type, and J is of contact type near $\partial\nu$. (One can for instance use the dotted concentric circles of Figure 3-30. We assume that $T \subset \nu$.) This pulls back to an almost complex structure on a tubular neighbourhood of $A \cup B$, compatible with ω . Moreover, we can extend it to an ω -compatible complex structure on the whole of $\mathcal{T}_{p,q,r}$, such that the extension also agrees with the usual complex structure outside a compact set. By convexity considerations, no holomorphic disc involved in the differential of the Floer cohomology between B and T can leave the tubular neighbourhood of $A \cup B$. In particular, the Floer cohomology in the local model agrees with the Floer cohomology in the total space.

Suppose we have a holomorphic map

$$\phi : \mathbb{R} \times [0, 1] \rightarrow \mathcal{C} \quad (3.5.18)$$

with Lagrangian boundary conditions given by T and B , and asymptotics b_1 and b_2 . First, by the open mapping theorem, we must have that $\chi \circ \phi = 0$. Thus the image of ϕ must lie in $\chi^{-1}(w)$. By topological considerations, the only possibilities are the two discs that we already

had. One can then check that they have Maslov index one, and, for instance using analogous arguments to the previous case, they are regular.

3.6 Mirror symmetry for $T_{p,q,r}$

This section contains the proof of Theorem 3.6.1, announced in the introduction:

Theorem 3.6.1. *There is an equivalence*

$$D^b Fuk^{\rightarrow}(T_{p,q,r}) \cong D^b Coh(\mathbb{P}_{p,q,r}^1) \quad (3.6.1)$$

where the left-hand side is the bounded derived directed Fukaya category of the singularity $T_{p,q,r}$, and the right-hand side is the bounded derived category of coherent sheaves on an orbifold \mathbb{P}^1 , with orbifold points of isotropies $1/p$, $1/q$ and $1/r$.

To obtain this result, we compare presentations of the two categories. The category of coherent sheaves on an orbifold \mathbb{P}^1 , and its bounded derived extension, were already understood. We shall use the same definitions as [17], and their description of the derived category $D^b Coh(\mathbb{P}_{p,q,r}^1)$ (see below). Using our description of $T_{p,q,r}$, we are able to calculate the bounded derived directed Fukaya category of $T_{p,q,r}$, and show that it is isomorphic to $D^b Coh(\mathbb{P}_{p,q,r}^1)$.

3.6.1 The derived category of coherent sheaves on an orbifold \mathbb{P}^1

We take the following description from [17, Section 6.9]: there is an equivalence

$$D^b Coh(\mathbb{P}_{p,q,r}^1) \cong D^b(\text{mod } \mathcal{A}) \quad (3.6.2)$$

where \mathcal{A} is the finite dimensional associative algebra given by the following quiver:

$$\begin{array}{c}
 \bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \bullet \\
 \begin{array}{l}
 \nearrow^{b_1} \bullet \xrightarrow{x_1} \bullet \xrightarrow{x_2} \dots \bullet \xrightarrow{x_{p-2}} \bullet \\
 \rightarrow^{b_2} \bullet \xrightarrow{y_1} \bullet \xrightarrow{y_2} \dots \bullet \xrightarrow{y_{q-2}} \bullet \\
 \searrow_{b_3} \bullet \xrightarrow{z_1} \bullet \xrightarrow{z_2} \dots \bullet \xrightarrow{z_{r-2}} \bullet
 \end{array}
 \end{array} \quad (3.6.3)$$

modulo the relations

$$b_1 \circ a_2 = 0 \quad b_2 \circ a_1 = 0 \quad b_3 \circ (a_1 - a_2) = 0 \quad (3.6.4)$$

(When comparing with [17, Section 6.9], note that we've assumed that the three orbifold points lie at $[0; 1]$, $[1; 0]$ and $[1; 1]$.) Note each vertex also comes with the identity morphism, which we suppress from the notation.

3.6.2 The derived directed Fukaya category of $T_{p,q,r}$

Given any singularity f , one can associate to it an A_∞ category, $D^b Fuk^{\rightarrow}(f)$. For a detailed introduction, see [62, Chapter 3]. Here's a definition. Suppose we have already fixed universal

choices of regular perturbation data to define the Fukaya category of the Milnor fibre of f . (This is the version of the Fukaya category that we have been considering so far.) Start with a distinguished basis of vanishing cycles for f , say V_1, \dots, V_μ . (Here we use not a cyclic ordering, but some fixed choice of absolute ordering that agrees with it.) The directed Fukaya category associated to our basis has objects the V_i . Morphism spaces are as follows:

$$\text{hom}(V_i, V_j) = \begin{cases} 0 & i > j \\ \mathbb{C}e_i & i = j \\ CF^*(V_i, V_j) & i < j \end{cases} \quad (3.6.5)$$

It is strictly unital. Suppose that $a_k \in CF(V_{k_1}, V_{k_2})$, with $k_1 < k_2$ for all k . Then the A_∞ -product

$$\mu^d(a_0, a_1, \dots, a_d) \quad (3.6.6)$$

is just given by the A_∞ -product in the Fukaya category of the Milnor fibre. In all other cases, apart from the μ^2 products implied by unitality, the A_∞ -products vanish. While each of the directed Fukaya categories depends a priori on the choice of distinguished basis, we get an invariant of the singularity when we pass to the bounded derived completion.

Let's compute this in the case of $T_{p,q,r}$. We'll use the geometric description 3.3.3, and the following ordered basis of vanishing cycles:

$$A, B, P_1, \dots, P_{p-1}, Q_1, \dots, Q_{q-1}, R_1, \dots, R_{r-1}. \quad (3.6.7)$$

(This can be obtained from the previous order through trivial mutations.) By general considerations, the A_∞ -category $D^b\mathcal{Fuk}^\rightarrow(T_{p,q,r})$ is isomorphic to $D^b(\text{mod}(\mathcal{B}))$, where \mathcal{B} is the endomorphism algebra of

$$A \oplus B \oplus P_1 \oplus \dots \oplus R_{r-1}.$$

We shall first describe this. What are the hom spaces between vanishing cycles? A and B intersect each of P_1, Q_1 and R_1 in one point, which lies on M_* . Label them as $p_A \in CF^*(A, P_1)$, and similarly with p_B, q_q, \dots, r_B in Figure 3-31. A and B intersect each other twice, giving generators $u, v \in CF^*(A, B)$. Aside from these, there is one intersection point between P_i and P_{i+1} , for each $i = 1, \dots, p-2$, say $p_i \in CF^*(P_i, P_{i+1})$, and similarly for the Q_i and R_i . There are no other intersection points between vanishing cycles.

To calculate \mathcal{B} , we need the A_∞ -morphisms between the vanishing cycles. We'll use the same symplectic form and almost complex structure as in Section 3.5. There are four holomorphic discs in Figure 3-31. Using arguments completely analogous to Section 3.5, we see that they are unique, of Maslov index zero, and regular. The intersections do not allow for A_∞ -morphisms between any other collection of vanishing cycles (even disregarding our choice of ordering). Thus the almost-complex structure we chose is regular. We have the following A_∞ -products:

$$\mu^2(p_B, u) = p_A \quad \mu^2(q_B, v) = q_A \quad \mu^2(r_B, u) = r_A \quad \mu^2(r_B, v) = r_A \quad (3.6.8)$$

(Note that we can always change signs of the generators so that these hold precisely.) Thus

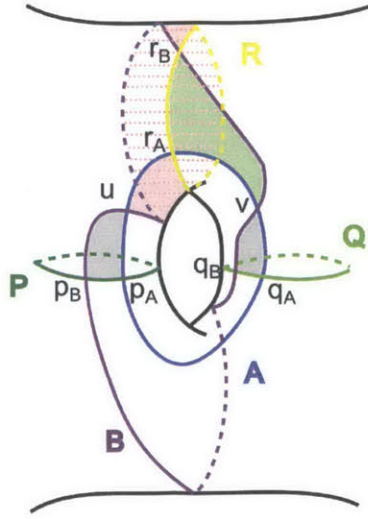


Figure 3-31: Fibre M_* , restricted to a neighbourhood of A and B .

\mathcal{B} is the finite dimensional associative algebra given by the directed quiver:

$$\begin{array}{ccccccc}
 & & & \bullet P_1 & \xrightarrow{p_1} & \bullet P_2 & \xrightarrow{p_2} & \dots & \xrightarrow{p_{p-2}} & \bullet P_{p-1} & & (3.6.9) \\
 & & & \nearrow^{p_B} & & & & & & & & \\
 \bullet A & \xrightleftharpoons[u]{u} & \bullet B & \xrightarrow{q_B} & \bullet Q_1 & \xrightarrow{q_1} & \bullet Q_2 & \xrightarrow{q_2} & \dots & \xrightarrow{q_{q-2}} & \bullet Q_{q-1} \\
 & & & \searrow_{r_B} & & & & & & & & \\
 & & & \bullet R_1 & \xrightarrow{r_1} & \bullet R_2 & \xrightarrow{r_2} & \dots & \xrightarrow{r_{r-2}} & \bullet R_{r-1} & &
 \end{array}$$

modulo the equations

$$p_B \circ v = 0 \quad q_B \circ u = 0 \quad r_B \circ (u - v) = 0 \quad (3.6.10)$$

and the fact that any sequence of the p_i , q_i or r_i composes to zero. (As with \mathcal{A} , we suppress the identity morphism at each vertex from the notation.) Now consider the following elements of $D^b \mathcal{Fuk}^\rightarrow(T_{p,q,r})$, given as twisted complexes:

$$P'_1 = \{P_1 \xrightarrow{p_1} P_2 \xrightarrow{p_2} \dots \xrightarrow{p_{p-2}} P_{p-1}\} \quad (3.6.11)$$

$$P'_2 = \{P_1 \xrightarrow{p_1} \dots \xrightarrow{p_{p-3}} P_{p-2}\} \quad (3.6.12)$$

$$\dots \quad P'_{p-1} = P_1 \quad (3.6.13)$$

and similarly for Q'_i and R'_i . Let Π_i be the projection $P'_{i+1} \rightarrow P'_i$. We'll use the same notation for the Q'_i and R'_i . The collection $A, B, P'_1, \dots, P'_{p-1}, Q'_1, \dots, R'_{r-1}$ also generates $D^b \mathcal{Fuk}^\rightarrow(T_{p,q,r})$. Also, each of the P'_i has one-dimensional self Floer cohomology. Thus the

endomorphism algebra of

$$A \oplus B \oplus P'_1 \oplus \dots \oplus R'_{r-1}$$

is precisely \mathcal{A} , given this time by the algebra of the directed quiver

$$\begin{array}{c}
 \bullet A \begin{array}{c} \xrightarrow{u} \\ \xleftarrow{v} \end{array} \bullet B \begin{array}{l} \nearrow^{p_B} \\ \xrightarrow{q_B} \\ \searrow_{r_B} \end{array} \begin{array}{c} \bullet P'_1 \xrightarrow{\Pi_2} \bullet P'_2 \xrightarrow{\Pi_3} \dots \xrightarrow{\Pi_{r-1}} \bullet P'_{r-1} \\ \bullet Q'_1 \xrightarrow{\Pi_2} \bullet Q'_2 \xrightarrow{\Pi_3} \dots \xrightarrow{\Pi_{q-1}} \bullet Q'_{q-1} \\ \bullet R'_1 \xrightarrow{\Pi_2} \bullet R'_2 \xrightarrow{\Pi_3} \dots \xrightarrow{\Pi_{r-1}} \bullet R'_{r-1} \end{array}
 \end{array} \tag{3.6.14}$$

modulo the equations $p_B \circ v = 0$, $q_B \circ u = 0$ and $r_B \circ (u - v) = 0$. Thus

$$D^b \mathcal{F}uk^{\rightarrow}(T_{p,q,r}) \cong D^b(\text{mod } \mathcal{A}).$$

This completes the proof of Theorem 3.6.1.

Appendix A

Visualizing the singular values of $M(x, y, z; t)$

The following code should allow the reader to visualise the singular values of $M(x, y, z; t)$ as t varies from 0 to 1. It was made and tested using Mathematica 8.0.

```
m[x_, y_] := -2((x + 0.25)^2 - 2 - 0.5 (y + 0.25)) ((y + 0.25)^2 - 2 -
    0.5 (x + 0.25))
u = 0;
g[x_] = Integrate[x (x + 1), x];
h[x_] = (8 I)*g[x];
plots1 = Table[
    g[x_, y_, z_] := m[x, y] - 2x*y + 2*(3 z + u*x*y)^2 + 2*h[z];
    spts = NSolve[D[g[x, y, z], x] == 0 && D[g[x, y, z], y] == 0 &&
        D[g[x, y, z], z] == 0, {x, y, z}];
    results = g[x, y, z] /. spts;
    ListPlot[{Re[#], Im[#]} & /@ results, AxesOrigin -> {0, 0},
        PlotRange -> {{-40, 20}, {-60, 20}}, ImagePadding -> 40,
        AspectRatio -> 1, Frame -> True,
        FrameLabel -> {{Im, None}, {Re, "complex plane"}},
        PlotStyle -> Directive[Red, PointSize[.02]]], {u, 0, 1, 0.02}];
ListAnimate[plots1]
```


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