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# Linear-Time Algorithm for Sliding Tokens on Trees

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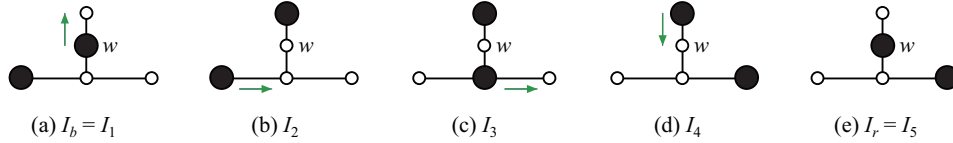
**Abstract.** Suppose that we are given two independent sets  $I_b$  and  $I_r$  of a graph such that  $|I_b| = |I_r|$ , and imagine that a token is placed on each vertex in  $I_b$ . Then, the SLIDING TOKEN problem is to determine whether there exists a sequence of independent sets which transforms  $I_b$  into  $I_r$  so that each independent set in the sequence results from the previous one by sliding exactly one token along an edge in the graph. This problem is known to be PSPACE-complete even for planar graphs, and also for bounded treewidth graphs. In this paper, we thus study the problem restricted to trees, and give the following three results: (1) the decision problem is solvable in linear time; (2) for a yes-instance, we can find in quadratic time an actual sequence of independent sets between  $I_b$  and  $I_r$  whose length (i.e., the number of token-slides) is quadratic; and (3) there exists an infinite family of instances on paths for which any sequence requires quadratic length.

## 1 Introduction

Recently, *reconfiguration problems* attract the attention in the field of theoretical computer science. The problem arises when we wish to find a step-by-step transformation between two feasible solutions of a problem such that all intermediate results are also feasible and each step abides by a fixed reconfiguration rule (i.e., an adjacency relation defined on feasible solutions of the original problem). This kind of reconfiguration problem has been studied extensively for several well-known problems, including INDEPENDENT SET [2, 5, 7, 10, 11, 13, 15, 19, 21–23], SATISFIABILITY [9, 20], SET COVER, CLIQUE, MATCHING [13], VERTEX-COLORING [3, 6, 8, 23], LIST EDGE-COLORING [14, 17], LIST  $L(2, 1)$ -LABELING [16], SUBSET SUM [12], SHORTEST PATH [4, 18], and so on.

### 1.1 SLIDING TOKEN

The SLIDING TOKEN problem was introduced by Hearn and Demaine [10] as a one-player game, which can be seen as a reconfiguration problem for INDEPENDENT SET. Recall that an *independent set* of a graph  $G$  is a vertex-subset of  $G$  in which no two vertices are adjacent. (Figure 1 depicts five different independent sets in the same graph.) Suppose that we are given two independent sets  $I_b$  and  $I_r$  of a graph  $G = (V, E)$  such that  $|I_b| = |I_r|$ , and imagine



**Fig. 1.** A sequence  $\langle I_1, I_2, \dots, I_5 \rangle$  of independent sets of the same graph, where the vertices in independent sets are depicted by large black circles (tokens).

that a token (coin) is placed on each vertex in  $I_b$ . Then, the SLIDING TOKEN problem is to determine whether there exists a sequence  $\langle I_1, I_2, \dots, I_\ell \rangle$  of independent sets of  $G$  such that

- (a)  $I_1 = I_b$ ,  $I_\ell = I_r$ , and  $|I_i| = |I_b| = |I_r|$  for all  $i$ ,  $1 \leq i \leq \ell$ ; and
- (b) for each  $i$ ,  $2 \leq i \leq \ell$ , there is an edge  $\{u, v\}$  in  $G$  such that  $I_{i-1} \setminus I_i = \{u\}$  and  $I_i \setminus I_{i-1} = \{v\}$ , that is,  $I_i$  can be obtained from  $I_{i-1}$  by sliding exactly one token on a vertex  $u \in I_{i-1}$  to its adjacent vertex  $v$  along  $\{u, v\} \in E$ .

Such a sequence is called a *reconfiguration sequence* between  $I_b$  and  $I_r$ . Figure 1 illustrates a reconfiguration sequence  $\langle I_1, I_2, \dots, I_5 \rangle$  of independent sets which transforms  $I_b = I_1$  into  $I_r = I_5$ . Hearn and Demaine proved that SLIDING TOKEN is PSPACE-complete for planar graphs, as an example of the application of their powerful tool, called the nondeterministic constraint logic model, which can be used to prove PSPACE-hardness of many puzzles and games [10], [11, Sec. 9.5].

## 1.2 Related and known results

As the (ordinary) INDEPENDENT SET problem is a key problem among thousands of NP-complete problems, SLIDING TOKEN plays a very important role since several PSPACE-hardness results have been proved using reductions from it. Indeed, SLIDING TOKEN is one of the most well-studied reconfiguration problems.

In addition, reconfiguration problems for INDEPENDENT SET (ISRECONF, for short) have been studied under different reconfiguration rules, as follows.

- *Token Sliding* (TS rule) [6, 7, 10, 11, 19, 23]: This rule corresponds to SLIDING TOKEN, that is, we can slide a single token only along an edge of a graph.
- *Token Jumping* (TJ rule) [7, 15, 19, 23]: A single token can “jump” to any vertex (including non-adjacent one) if it results in an independent set.
- *Token Addition and Removal* (TAR rule) [2, 5, 13, 19, 21–23]: We can either add or remove a single token at a time if it results in an independent set of cardinality at least a given threshold minus one. Therefore, under the TAR rule, independent sets in the sequence do not have the same cardinality.

We note that the existence of a desired sequence depends deeply on the reconfiguration rules. (See Fig. 2 for example.) However, ISRECONF is PSPACE-complete under any of the three reconfiguration rules for planar graphs [6, 10, 11], for perfect graphs [19], and for bounded bandwidth graphs [23]. The PSPACE-hardness implies that, unless  $\text{NP} = \text{PSPACE}$ , there exists an instance of SLIDING TOKEN which requires a super-polynomial number of token-slides even in a minimum-length reconfiguration sequence. In such a case, tokens should



**Fig. 2.** A yes-instance for ISRECONF under the TJ rule, which is a no-instance for the SLIDING TOKEN problem.

make “detours” to avoid violating independence. (For example, see the token placed on the vertex  $w$  in Fig. 1(a); it is moved twice even though  $w \in I_b \cap I_r$ .)

We here explain only the results which are strongly related to this paper, that is, SLIDING TOKEN on trees; see the references above for the other results.

### Results for TS rule (SLIDING TOKEN).

Kamiński et al. [19] gave a linear-time algorithm to solve SLIDING TOKEN for cographs (also known as  $P_4$ -free graphs). They also showed that, for any yes-instance on cographs, two given independent sets  $I_b$  and  $I_r$  have a reconfiguration sequence such that no token makes detour.

Very recently, Bonsma et al. [7] proved that SLIDING TOKEN can be solved in polynomial time for claw-free graphs. Note that neither cographs nor claw-free graphs contain trees as a (proper) subclass. Thus, the complexity status for trees was open under the TS rule.

### Results for trees.

In contrast to the TS rule, it is known that ISRECONF can be solved in linear time under the TJ and TAR rules for even-hole-free graphs [19], which include trees. Indeed, the answer is always “yes” under the two rules when restricted to even-hole-free graphs. Furthermore, tokens never make detours in even-hole-free graphs under the TJ and TAR rules.

On the other hand, under the TS rule, tokens are required to make detours even in trees. (See Fig. 1.) In addition, there are no-instances for trees under TS rule. (See Fig. 2.) These make the problem much more complicated, and we think they are the main reasons why SLIDING TOKEN for trees was open, despite the recent intensive algorithmic research on ISRECONF [2, 5, 7, 15, 19, 22].

## 1.3 Our contribution

In this paper, we first prove that the SLIDING TOKEN problem is solvable in  $O(n)$  time for any tree  $T$  with  $n$  vertices. Therefore, we can conclude that ISRECONF for trees is in P (indeed, solvable in linear time) under any of the three reconfiguration rules.

It is remarkable that there exists an infinite family of instances on paths for which any reconfiguration sequence requires  $\Omega(n^2)$  length, although we can decide it is a yes-instance in  $O(n)$  time. As the second result of this paper, we give an  $O(n^2)$ -time algorithm which finds an actual reconfiguration sequence of length  $O(n^2)$  between two given independent sets for a yes-instance.

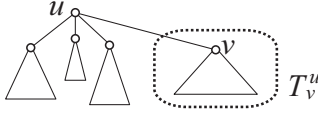
Since the treewidth of any graph  $G$  can be bounded by the bandwidth of  $G$ , the result of [23] implies that SLIDING TOKEN is PSPACE-complete for bounded treewidth graphs. (See [1] for the definition of treewidth.) Thus, there exists an instance on bounded treewidth graphs which requires a super-polynomial number of token-slides even in a minimum-length reconfiguration sequence unless  $\text{NP} = \text{PSPACE}$ . Therefore, it is interesting that any yes-instance on a tree, whose treewidth is one, has an  $O(n^2)$ -length reconfiguration sequence even though trees require to make detours to transform.

## 1.4 Technical overview

We here explain our main ideas; formal descriptions will be given later.

We say that a token on a vertex  $v$  is “rigid” under an independent set  $I$  of a tree  $T$  if it cannot be slid at all, that is,  $v \in I'$  holds for *any* independent set  $I'$  of  $T$  which is reconfigurable from  $I$ . (For example, four tokens in Fig. 2 are rigid.) Our algorithm is based on the following two key points.

- (1) In Lemma 1, we will give a simple but non-trivial characterization of rigid tokens, based on which we can find all rigid tokens of two given independent sets  $I_b$  and  $I_r$  in time  $O(n)$ . Note that, if  $I_b$  and  $I_r$  have different placements of rigid tokens, then it is a no-instance (Lemma 5).



**Fig. 3.** Subtree  $T_v^u$  in the whole tree  $T$ .

- (2) Otherwise, we obtain a forest by deleting the vertices with rigid tokens together with their neighbors (Lemma 6). We will prove in Lemma 7 that the answer is “yes” as long as each tree in the forest contains the same number of tokens in  $I_b$  and  $I_r$ .

## 2 Preliminaries

In this section, we introduce some basic terms and notation.

### 2.1 Graph notation

In the SLIDING TOKEN problem, we may assume without loss of generality that graphs are simple and connected. For a graph  $G$ , we sometimes denote by  $V(G)$  and  $E(G)$  the vertex set and edge set of  $G$ , respectively.

In a graph  $G$ , a vertex  $w$  is said to be a *neighbor* of a vertex  $v$  if  $\{v, w\} \in E(G)$ . For a vertex  $v$  in  $G$ , let  $N(G, v) = \{w \in V(G) \mid \{v, w\} \in E(G)\}$ , and let  $N[G, v] = N(G, v) \cup \{v\}$ . For a subset  $S \subseteq V(G)$ , we simply write  $N[G, S] = \bigcup_{v \in S} N[G, v]$ . For a vertex  $v$  of  $G$ , we denote by  $\deg_G(v)$  the degree of  $v$  in  $G$ , that is,  $\deg_G(v) = |N(G, v)|$ . For a subgraph  $G'$  of a graph  $G$ , we denote by  $G \setminus G'$  the subgraph of  $G$  induced by the vertices in  $V(G) \setminus V(G')$ .

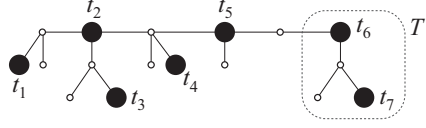
Let  $T$  be a tree. For two vertices  $v$  and  $w$  in  $T$ , the unique path between  $v$  and  $w$  is simply called the  *$vw$ -path* in  $T$ . We denote by  $\text{dist}(v, w)$  the number of edges in the  $vw$ -path in  $T$ . For two vertices  $u$  and  $v$  of a tree  $T$ , let  $T_v^u$  be the subtree of  $T$  obtained by regarding  $u$  as the root of  $T$  and then taking the subtree rooted at  $v$  which consists of  $v$  and all descendants of  $v$ . (See Fig. 3.) It should be noted that  $u$  is not contained in the subtree  $T_v^u$ .

### 2.2 Definitions for SLIDING TOKEN

Let  $I_i$  and  $I_j$  be two independent sets of a graph  $G$  such that  $|I_i| = |I_j|$ . If there exists exactly one edge  $\{u, v\}$  in  $G$  such that  $I_i \setminus I_j = \{u\}$  and  $I_j \setminus I_i = \{v\}$ , then we say that  $I_j$  can be obtained from  $I_i$  by *sliding* the token on  $u \in I_i$  to its adjacent vertex  $v$  along the edge  $\{u, v\}$ , and denote it by  $I_i \leftrightarrow I_j$ . We note that the tokens are unlabeled, while the vertices in a graph are labeled. We sometimes omit to say (the label of) the vertex on which a token is placed, and simply say “a token in an independent set  $I$ .”

A *reconfiguration sequence* between two independent sets  $I_1$  and  $I_\ell$  of  $G$  is a sequence  $\langle I_1, I_2, \dots, I_\ell \rangle$  of independent sets of  $G$  such that  $I_{i-1} \leftrightarrow I_i$  for  $i = 2, 3, \dots, \ell$ . We sometimes write  $I \in \mathcal{S}$  if an independent set  $I$  of  $G$  appears in the reconfiguration sequence  $\mathcal{S}$ . We write  $I_1 \overset{G}{\rightsquigarrow} I_\ell$  if there exists a reconfiguration sequence  $\mathcal{S}$  between  $I_1$  and  $I_\ell$  such that all independent sets  $I \in \mathcal{S}$  satisfy  $I \subseteq V(G)$ . The *length* of a reconfiguration sequence  $\mathcal{S}$  is defined as the number of independent sets contained in  $\mathcal{S}$ . For example, the length of the reconfiguration sequence in Fig. 1 is 5.

Given two independent sets  $I_b$  and  $I_r$  of a graph  $G$ , the SLIDING TOKEN problem is to determine whether  $I_b \overset{G}{\rightsquigarrow} I_r$  or not. We may assume without loss of generality that  $|I_b| = |I_r|$ ; otherwise the answer is clearly “no.” Note that SLIDING TOKEN is a decision problem asking for the existence of a reconfiguration sequence between  $I_b$  and  $I_r$ , and hence it does not ask for an actual reconfiguration sequence. We always denote by  $I_b$  and  $I_r$  the *initial* and *target* independent sets of  $G$ , respectively.



**Fig. 4.** An independent set  $I$  of a tree  $T$ , where  $t_1, t_2, t_3, t_4$  are  $(T, I)$ -rigid tokens and  $t_5, t_6, t_7$  are  $(T, I)$ -movable tokens. For the subtree  $T'$ , tokens  $t_6, t_7$  are  $(T', I \cap T')$ -rigid.

### 3 Algorithm for Trees

In this section, we give the main result of this paper.

**Theorem 1.** *The SLIDING TOKEN problem can be solved in linear time for trees.*

As a proof of Theorem 1, we give an  $O(n)$ -time algorithm which solves SLIDING TOKEN for a tree with  $n$  vertices.

#### 3.1 Rigid tokens

In this subsection, we formally define the concept of rigid tokens, and give their nice characterization.

Let  $T$  be a tree, and let  $I$  be an independent set of  $T$ . We say that a token on a vertex  $v \in I$  is  $(T, I)$ -rigid if  $v \in I'$  holds for *any* independent set  $I'$  of  $T$  such that  $I \overset{T}{\rightsquigarrow} I'$ . Conversely, if a token on a vertex  $v \in I$  is not  $(T, I)$ -rigid, then it is  $(T, I)$ -movable; in other words, there exists an independent set  $I'$  such that  $v \notin I'$  and  $I \overset{T}{\rightsquigarrow} I'$ . For example, in Fig. 4, the tokens  $t_1, t_2, t_3, t_4$  are  $(T, I)$ -rigid, while the tokens  $t_5, t_6, t_7$  are  $(T, I)$ -movable. Note that, even though  $t_6$  and  $t_7$  cannot be slid to any neighbor in  $T$  under  $I$ , we can slide them after sliding  $t_5$  downward.

We then extend the concept of rigid/movable tokens to subtrees of  $T$ . For any subtree  $T'$  of  $T$ , we denote simply  $I \cap T' = I \cap V(T')$ . Then, a token on a vertex  $v \in I \cap T'$  is  $(T', I \cap T')$ -rigid if  $v \in J$  holds for *any* independent set  $J$  of  $T'$  such that  $I \cap T' \overset{T'}{\rightsquigarrow} J$ . For example, in Fig. 4, tokens  $t_6$  and  $t_7$  are  $(T', I \cap T')$ -rigid even though they are  $(T, I)$ -movable in the whole tree  $T$ . Note that, since independent sets are restricted only to the subtree  $T'$ , we cannot use any vertex (and hence any edge) in  $T \setminus T'$  during the reconfiguration. Furthermore, the vertex-subset  $J \cup (I \cap (T \setminus T'))$  does not necessarily form an independent set of the whole tree  $T$ .

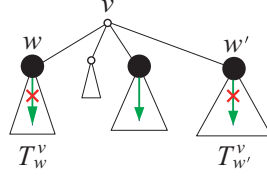
We now give our first key lemma, which gives a characterization of rigid tokens. (See also Fig. 5(a) for the claim (b) below.)

**Lemma 1.** *Let  $I$  be an independent set of a tree  $T$ , and let  $u$  be a vertex in  $I$ .*

(a) *Suppose that  $|V(T)| = |\{u\}| = 1$ . Then, the token on  $u$  is  $(T, I)$ -rigid.*



**Fig. 5.** (a) A  $(T, I)$ -rigid token on  $u$ , and (b) a  $(T, I)$ -movable token on  $u$ .



**Fig. 6.** Illustration for Lemma 2.

- (b) Suppose that  $|V(T)| \geq 2$ . Then, a token on  $u$  is  $(T, I)$ -rigid if and only if, for every neighbor  $v \in N(T, u)$ , there exists a vertex  $w \in I \cap N(T_v^u, v)$  such that the token on  $w$  is  $(T_w^v, I \cap T_w^v)$ -rigid.

*Proof.* Obviously, the claim (a) holds. In the following, we thus assume that  $|V(T)| \geq 2$  and prove the claim (b).

We first show the if-part. Suppose that, for every neighbor  $v \in N(T, u)$ , there exists a vertex  $w \in I \cap N(T_v^u, v)$  such that the token on  $w$  is  $(T_w^v, I \cap T_w^v)$ -rigid. (See Fig. 5(a).) Then, we will prove that the token  $t$  on  $u$  is  $(T, I)$ -rigid. Since we can slide a token only along an edge of  $T$ , if  $t$  is not  $(T, I)$ -rigid (and hence is  $(T, I)$ -movable), then it must be slid to some neighbor  $v \in N(T, u)$ . By the assumption,  $v$  is adjacent with another token  $t'$  placed on  $w \in I \cap N(T_v^u, v)$ , and hence we first have to slide  $t'$  to one of its neighbors other than  $v$ . However, this is impossible since the token  $t'$  on  $w$  is assumed to be  $(T_w^v, I \cap T_w^v)$ -rigid and hence  $w \in J$  holds for any independent set  $J$  of  $T_w^v$  such that  $I \cap T_w^v \overset{T_w^v}{\rightsquigarrow} J$ . We can thus conclude that  $t$  is  $(T, I)$ -rigid.

We then show the only-if-part by taking a contrapositive. Suppose that  $u$  has a neighbor  $v \in N(T, u)$  such that either  $I \cap N(T_v^u, v) = \emptyset$  or all tokens on  $w \in I \cap N(T_v^u, v)$  are  $(T_w^v, I \cap T_w^v)$ -movable. (See Fig. 5(b).) Then, we will prove that the token  $t$  on  $u$  is  $(T, I)$ -movable; in particular, we can slide  $t$  from  $u$  to  $v$ . Since any token  $t'$  on a vertex  $w \in I \cap N(T_v^u, v)$  is  $(T_w^v, I \cap T_w^v)$ -movable, we can slide  $t'$  to some vertex in  $T_w^v$  via a reconfiguration sequence  $\mathcal{S}_w$  in  $T_w^v$ . Recall that only the vertex  $v$  is adjacent with a vertex in  $T_w^v$  and  $v \notin I$ . Therefore,  $\mathcal{S}_w$  can be naturally extended to a reconfiguration sequence  $\mathcal{S}$  in the whole tree  $T$  such that  $I' \cap (T \setminus T_w^v) = I \cap (T \setminus T_w^v)$  holds for any independent set  $I' \in \mathcal{S}$  of  $T$ . Apply this process to all tokens on vertices in  $I \cap N(T_v^u, v)$ , and obtain an independent set  $I''$  of  $T$  such that  $I'' \cap N(T_v^u, v) = \emptyset$ . Then, we can slide the token  $t$  on  $u$  to  $v$ . Thus,  $t$  is  $(T, I)$ -movable.  $\square$

The following lemma is useful for proving the correctness of our algorithm in Section 3.3.

**Lemma 2.** Let  $I$  be an independent set of a tree  $T$  such that all tokens are  $(T, I)$ -movable, and let  $v$  be a vertex such that  $v \notin I$ . Then, there exists at most one neighbor  $w \in I \cap N(T, v)$  such that the token on  $w$  is  $(T_w^v, I \cap T_w^v)$ -rigid.

*Proof.* Suppose for a contradiction that there exist two neighbors  $w$  and  $w'$  in  $I \cap N(T, v)$  such that the tokens on  $w$  and  $w'$  are  $(T_w^v, I \cap T_w^v)$ -rigid and  $(T_{w'}^v, I \cap T_{w'}^v)$ -rigid, respectively. (See Fig. 6.) Since the token  $t$  on  $w$  is  $(T_w^v, I \cap T_w^v)$ -rigid but is  $(T, I)$ -movable, there is a reconfiguration sequence  $\mathcal{S}_t$  starting from  $I$  which slides  $t$  to  $v$ . However, before sliding  $t$  to  $v$ ,  $\mathcal{S}_t$  must slide the token  $t'$  on  $w'$  to some vertex in  $N(T_w^v, w')$ . This contradicts the assumption that  $t'$  is  $(T_{w'}^v, I \cap T_{w'}^v)$ -rigid.  $\square$

### 3.2 Linear-time algorithm

In this subsection, we describe an algorithm to solve the SLIDING TOKEN problem for trees, and estimate its running time; the correctness of the algorithm will be proved in Section 3.3.

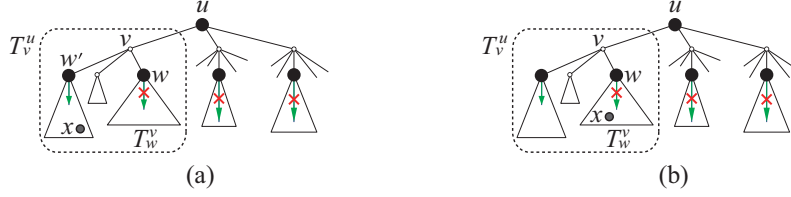


Fig. 7. Illustration for Lemma 3.

Let  $T$  be a tree with  $n$  vertices, and let  $I_b$  and  $I_r$  be two given independent sets of  $T$ . For an independent set  $I$  of  $T$ , we denote by  $R(I)$  the set of all vertices in  $I$  on which  $(T, I)$ -rigid tokens are placed. Then, the following algorithm determines whether  $I_b \stackrel{T}{\leftrightarrow} I_r$  or not.

- Step 1.** Compute  $R(I_b)$  and  $R(I_r)$ . Return “no” if  $R(I_b) \neq R(I_r)$ ; otherwise go to Step 2.
- Step 2.** Delete the vertices in  $N[T, R(I_b)] = N[T, R(I_r)]$  from  $T$ , and obtain a forest  $F$  consisting of  $q$  trees  $T_1, T_2, \dots, T_q$ . Return “yes” if  $|I_b \cap T_j| = |I_r \cap T_j|$  holds for every  $j \in \{1, 2, \dots, q\}$ ; otherwise return “no.”

We now show that our algorithm above runs in  $O(n)$  time. Clearly, Step 2 can be done in  $O(n)$  time, and hence we will show that Step 1 can be executed in  $O(n)$  time.

We first give the following property of rigid tokens on a tree, which says that deleting movable tokens does not affect the rigidity of the other tokens.

**Lemma 3.** *Let  $I$  be an independent set of a tree  $T$ . Assume that the token on a vertex  $x \in I$  is  $(T, I)$ -movable. Then, for every vertex  $u \in I \setminus \{x\}$ , the token on  $u$  is  $(T, I)$ -rigid if and only if it is  $(T, I \setminus \{x\})$ -rigid.*

*Proof.* The if-part is trivially true, because we cannot make a rigid token movable by adding another token. We thus show the only-if-part by contradiction.

Let  $I' = I \setminus \{x\}$ . Suppose that  $u \in I$  is the closest vertex to  $x$  such that its token is  $(T, I)$ -rigid but  $(T, I')$ -movable. We assume that  $x$  is contained in a subtree  $T_v^u$  for a neighbor  $v$  of  $u$ . (See Fig. 7.) Note that  $x \neq v$  since  $x, u \in I$ . Since the token  $t_u$  on  $u$  is  $(T, I)$ -rigid, by Lemma 1 the vertex  $v \in N(T, u)$  has at least one neighbor  $w \in I \cap N(T_v^u, v)$  such that the token  $t_w$  on  $w$  is  $(T_w^v, I \cap T_w^v)$ -rigid. Indeed,  $t_w$  is  $(T, I)$ -rigid, because  $t_u$  is assumed to be  $(T, I)$ -rigid. Thus, we know that  $x \neq w$  since the token  $t_x$  on  $x$  is  $(T, I)$ -movable.

First, consider the case where  $x$  is contained in a subtree  $T_{w'}^v$  for some neighbor  $w'$  of  $v$  other than  $w$ . (See Fig. 7(a).) Then,  $I' \cap T_w^v = I \cap T_w^v$ . Since  $t_w$  is  $(T_w^v, I \cap T_w^v)$ -rigid, it is also  $(T_w^v, I' \cap T_w^v)$ -rigid. Therefore, by Lemma 1 the token  $t_u$  is  $(T, I')$ -rigid. This contradicts the assumption that  $t_u$  is  $(T, I')$ -movable.

We thus consider the case where  $x \in V(T_w^v) \setminus \{w\}$ . (See Fig. 7(b).) Recall that  $I'$  is obtained by deleting only  $x$  from  $I$ . Then, since  $t_u$  is  $(T, I)$ -rigid but  $(T, I')$ -movable, it must be slid from  $u$  to  $v$ . However, before executing this token-slide, we have to slide  $t_w$  to some vertex in  $N(T_w^v, w)$ . Thus,  $t_w$  is  $(T_w^v, I' \cap T_w^v)$ -movable, and hence it is also  $(T, I')$ -movable. Since  $t_w$  is  $(T, I)$ -rigid and  $w$  is strictly closer to  $x \in V(T_w^v)$  than  $u$ , this contradicts the assumption that  $u$  is the closest vertex to  $x$  such that its token is  $(T, I)$ -rigid but  $(T, I')$ -movable.  $\square$

Then, the following lemma proves that Step 1 can be executed in  $O(n)$  time.

**Lemma 4.** *For an independent set  $I$  of a tree  $T$  with  $n$  vertices,  $R(I)$  can be computed in  $O(n)$  time.*

*Proof.* Lemma 3 implies that the set  $R(I)$  of all  $(T, I)$ -rigid tokens in  $I$  can be found by removing all  $(T, I)$ -movable tokens in  $I$ . Observe that, if  $I$  contains  $(T, I)$ -movable tokens,

then at least one of them can be immediately slid to one of its neighbors. That is, there is a token on  $u \in I$  which has a neighbor  $w \in N(T, u)$  such that  $N(T, w) \cap I = \{u\}$ . Then, the following algorithm efficiently finds and removes such tokens iteratively.

**Step A.** Define and compute  $\deg_I(w) = |N(T, w) \cap I|$  for all vertices  $w \in V(T)$ .

**Step B.** Define and compute  $M = \{u \in I \mid \exists w \in N(T, u) \text{ such that } \deg_I(w) = 1\}$ .

**Step C.** Repeat the following steps (i)–(iii) until  $M = \emptyset$ .

- (i) Select an arbitrary vertex  $u \in M$ , and remove it from  $M$  and  $I$ .
- (ii) Update  $\deg_I(w) := \deg_I(w) - 1$  for each neighbor  $w \in N(T, u)$ .
- (iii) If  $\deg_I(w)$  becomes one by the update (ii) above, then add the vertex  $u' \in N(T, w) \cap I$  into  $M$ .

**Step D.** Output  $I$  as the set  $R(I)$ .

Clearly, Steps A, B and D can be done in  $O(n)$  time. We now show that Step C takes only  $O(n)$  time. Each vertex in  $I$  can be selected at most once as  $u$  at Step C-(i). For the selected vertex  $u$ , Step C-(ii) takes  $O(\deg_T(u))$  time for updating  $\deg_I(w)$  of its neighbors  $w \in N(T, u)$ . Each vertex in  $V(T) \setminus I$  can be selected at most once as  $w$  at Step C-(iii). For the selected vertex  $w$ , Step C-(iii) takes  $O(\deg_T(w))$  time for finding  $u' \in N(T, w) \cap I$ . Therefore, Step C takes  $O\left(\sum_{v \in V(T)} \deg_T(v)\right) = O(n)$  time in total.  $\square$

Therefore, Step 1 of our algorithm can be done in  $O(n)$  time, and hence the algorithm runs in linear time in total.

### 3.3 Correctness of the algorithm

In this subsection, we prove that the  $O(n)$ -time algorithm in Section 3.2 correctly determines whether  $I_b \overset{T}{\rightsquigarrow} I_r$  or not, for two given independent sets  $I_b$  and  $I_r$  of a tree  $T$ .

We first show the correctness of Step 1.

**Lemma 5.** *Suppose that  $R(I_b) \neq R(I_r)$  for two given independent sets  $I_b$  and  $I_r$  of a tree  $T$ . Then, it is a no-instance.*

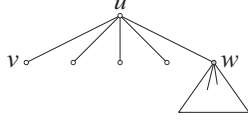
*Proof.* By the definition of rigid tokens,  $R(I_b) = R(I')$  holds for *any* independent set  $I'$  of  $T$  such that  $I_b \overset{T}{\rightsquigarrow} I'$ . Therefore, there is no reconfiguration sequence between  $I_b$  and  $I_r$  if  $R(I_r) \neq R(I_b)$ .  $\square$

We then show the correctness of Step 2. We first claim that deleting the vertices with rigid tokens together with their neighbors does not affect the reconfigurability.

**Lemma 6.** *Suppose that  $R(I_b) = R(I_r)$  for two given independent sets  $I_b$  and  $I_r$  of a tree  $T$ , and let  $F$  be the forest obtained by deleting the vertices in  $N[T, R(I_b)] = N[T, R(I_r)]$  from  $T$ . Then,  $I_b \overset{T}{\rightsquigarrow} I_r$  if and only if  $I_b \cap F \overset{F}{\rightsquigarrow} I_r \cap F$ . Furthermore, all tokens in  $I_b \cap F$  are  $(F, I_b \cap F)$ -movable, and all tokens in  $I_r \cap F$  are  $(F, I_r \cap F)$ -movable.*

*Proof.* We first prove the if-part. Suppose that  $I_b \cap F \overset{F}{\rightsquigarrow} I_r \cap F$ , and hence there exists a reconfiguration sequence  $\mathcal{S}_F$  between  $I_b \cap F$  and  $I_r \cap F$ . Then, for each independent set  $I \in \mathcal{S}_F$  of  $F$ , the vertex-subset  $R(I_b) \cup I = R(I_r) \cup I$  forms an independent set of  $T$  since  $F$  is obtained by deleting all vertices in  $N[T, R(I_b)] = N[T, R(I_r)]$ . Therefore,  $\mathcal{S}_F$  can be extended to a reconfiguration sequence between  $I_b$  and  $I_r$  of  $T$ . We thus have  $I_b \overset{T}{\rightsquigarrow} I_r$ .

We then prove the only-if-part. Suppose that  $I_b \overset{T}{\rightsquigarrow} I_r$ , and hence there exists a reconfiguration sequence  $\mathcal{S}_T$  between  $I_b$  and  $I_r$ . Then, for any independent set  $I \in \mathcal{S}_T$ , we have  $I_b \overset{T}{\rightsquigarrow} I$  and  $I \overset{T}{\rightsquigarrow} I_r$ , and hence by the definition of rigid tokens  $R(I_b) = R(I_r) \subseteq I$  holds. Furthermore,  $I \setminus R(I_b) = I \setminus R(I_r)$  is a vertex-subset of  $V(F)$  since no token can be placed



**Fig. 8.** A degree-1 vertex  $v$  of a tree  $T$  which is safe.

on any neighbor of  $R(I_b) = R(I_r)$ . Therefore,  $I \setminus R(I_b) = I \setminus R(I_r)$  forms an independent set of  $F$ . For two consecutive independent sets  $I_{i-1}$  and  $I_i$  in  $\mathcal{S}_T$ , let  $I_{i-1} \setminus I_i = \{u\}$  and  $I_i \setminus I_{i-1} = \{v\}$ . Since  $u \notin I_i$  and  $v \notin I_{i-1}$ , neither  $u$  nor  $v$  are in  $R(I_b) = R(I_r)$ . Therefore, we have  $u, v \in V(F)$ , and hence the edge  $\{u, v\}$  is in  $E(F)$ . Then, we can obtain a reconfiguration sequence between  $I_b \cap F$  and  $I_r \cap F$  by replacing all independent sets  $I \in \mathcal{S}_T$  with  $I \cap F$ . We thus have  $I_b \cap F \overset{F}{\rightsquigarrow} I_r \cap F$ .

We finally prove that all tokens in  $I_b \cap F$  are  $(F, I_b \cap F)$ -movable. (The proof for the tokens in  $I_r \cap F$  is the same.) Notice that each token  $t$  on a vertex  $v$  in  $I_b \cap F$  is  $(T, I_b)$ -movable; otherwise  $t \in R(I_b)$ . Therefore, there exists an independent set  $I'$  of  $T$  such that  $v \notin I'$  and  $I_b \overset{T}{\rightsquigarrow} I'$ . Then,  $I_b \cap F \overset{F}{\rightsquigarrow} I' \cap F$  as we have proved above, and hence  $t$  is  $(F, I_b \cap F)$ -movable.  $\square$

Suppose that  $R(I_b) = R(I_r)$  for two given independent sets  $I_b$  and  $I_r$  of a tree  $T$ . Let  $F$  be the forest consisting of  $q$  trees  $T_1, T_2, \dots, T_q$ , which is obtained from  $T$  by deleting the vertices in  $N[T, R(I_b)] = N[T, R(I_r)]$ . Since we can slide a token only along an edge of  $F$ , we clearly have  $I_b \cap F \overset{F}{\rightsquigarrow} I_r \cap F$  if and only if  $I_b \cap T_j \overset{T_j}{\rightsquigarrow} I_r \cap T_j$  for all  $j \in \{1, 2, \dots, q\}$ . Furthermore, Lemma 6 implies that, for each  $j \in \{1, 2, \dots, q\}$ , all tokens in  $I_b \cap T_j$  are  $(T_j, I_b \cap T_j)$ -movable; similarly, all tokens in  $I_r \cap T_j$  are  $(T_j, I_r \cap T_j)$ -movable.

We now give our second key lemma, which completes the correctness proof of our algorithm.

**Lemma 7.** *Let  $I_b$  and  $I_r$  be two independent sets of a tree  $T$  such that all tokens in  $I_b$  and  $I_r$  are  $(T, I_b)$ -movable and  $(T, I_r)$ -movable, respectively. Then,  $I_b \overset{T}{\rightsquigarrow} I_r$  if and only if  $|I_b| = |I_r|$ .*

The only-if-part of Lemma 7 is trivial, and hence we prove the if-part. In our proof, we do *not* reconfigure  $I_b$  into  $I_r$  directly, but reconfigure both  $I_b$  and  $I_r$  into some independent set  $I^*$  of  $T$ . Note that, since any reconfiguration sequence is reversible,  $I_b \overset{T}{\rightsquigarrow} I^*$  and  $I_r \overset{T}{\rightsquigarrow} I^*$  imply that  $I_b \overset{T}{\rightsquigarrow} I_r$ .

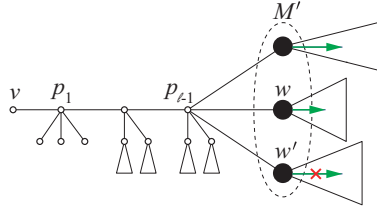
We say that a degree-1 vertex  $v$  of  $T$  is *safe* if its unique neighbor  $u$  has at most one neighbor  $w$  of degree more than one. (See Fig. 8.) Note that any tree has at least one safe degree-1 vertex.

As the first step of the if-part proof, we give the following lemma.

**Lemma 8.** *Let  $I$  be an independent set of a tree  $T$  such that all tokens in  $I$  are  $(T, I)$ -movable, and let  $v$  be a safe degree-1 vertex of  $T$ . Then, there exists an independent set  $I'$  such that  $v \in I'$  and  $I \overset{T}{\rightsquigarrow} I'$ .*

*Proof.* Suppose that  $v \notin I$ ; otherwise the lemma clearly holds. We will show that one of the closest tokens from  $v$  can be slid to  $v$ . Let  $M = \{w \in I \mid \text{dist}(v, w) = \min_{x \in I} \text{dist}(v, x)\}$ . Let  $w$  be an arbitrary vertex in  $M$ , and let  $P = (p_0 = v, p_1, \dots, p_\ell = w)$  be the  $vw$ -path in  $T$ . (See Fig. 9.) If  $\ell = 1$  and hence  $p_1 \in I$ , then we can simply slide the token on  $p_1$  to  $v$ . Thus, we may assume that  $\ell \geq 2$ .

We note that no token is placed on the vertices  $p_0, \dots, p_{\ell-1}$  and the neighbors of  $p_0, \dots, p_{\ell-2}$ , because otherwise the token on  $w$  is not closest to  $v$ . Let  $M' = M \cap N(T, p_{\ell-1})$ .



**Fig. 9.** Illustration for Lemma 8.

Since  $p_{\ell-1} \notin I$ , by Lemma 2 there exists at most one vertex  $w' \in M'$  such that the token on  $w'$  is  $(T_{w'}^{p_{\ell-1}}, I \cap T_{w'}^{p_{\ell-1}})$ -rigid. We choose such a vertex  $w'$  if it exists, otherwise choose an arbitrary vertex in  $M'$  and regard it as  $w'$ .

Since all tokens on the vertices  $w''$  in  $M' \setminus \{w'\}$  are  $(T_{w''}^{p_{\ell-1}}, I \cap T_{w''}^{p_{\ell-1}})$ -movable, we first slide the tokens on  $w''$  to some vertices in  $T_{w''}^{p_{\ell-1}}$ . Then, we can slide the token on  $w'$  to  $v$  along the path  $P$ . In this way, we can obtain an independent set  $I'$  such that  $v \in I'$  and  $I \overset{T}{\rightsquigarrow} I'$ .  $\square$

We then prove that deleting a safe degree-1 vertex with a token does not affect the movability of the other tokens. (See also Fig. 10.)

**Lemma 9.** *Let  $v$  be a safe degree-1 vertex of a tree  $T$ , and let  $\bar{T}$  be the subtree of  $T$  obtained by deleting  $v$ , its unique neighbor  $u$ , and the resulting isolated vertices. Let  $I$  be an independent set of  $T$  such that  $v \in I$  and all tokens are  $(T, I)$ -movable. Then, all tokens in  $I \setminus \{v\}$  are  $(\bar{T}, I \setminus \{v\})$ -movable.*

*Proof.* Since  $T_v^u$  consists of a single vertex  $v$ , the token on  $v$  is  $(T_v^u, I \cap T_v^u)$ -rigid. Therefore, no token is placed on degree-1 neighbors of  $u$  other than  $v$  (see Fig. 10), because otherwise it contradicts to Lemma 2; recall that all tokens in  $I$  are assumed to be  $(T, I)$ -movable.

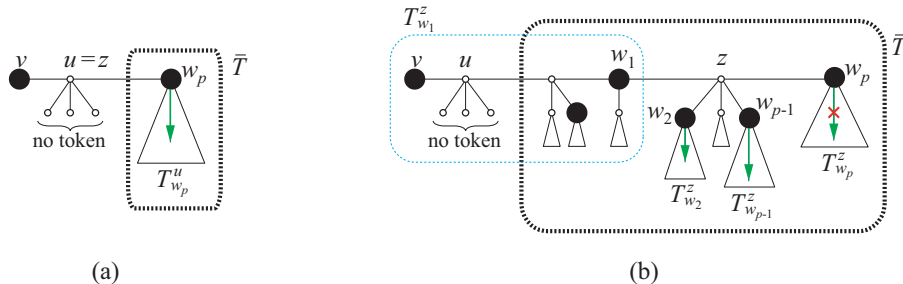
Let  $\bar{I} = I \setminus \{v\}$ . Suppose for a contradiction that there exists a token in  $\bar{I}$  which is  $(\bar{T}, \bar{I})$ -rigid. Let  $w_p \in \bar{I}$  be such a vertex closest to  $v$ , and let  $z$  be the vertex on the  $vw_p$ -path right before  $w_p$ .

**Case (1):**  $z = u$ . (See Fig. 10(a).)

Recall that the token on  $v$  is  $(T, I)$ -movable, but is  $(T_v^u, I \cap T_v^u)$ -rigid. Therefore, by Lemma 2 the token on  $w_p$  must be  $(T_{w_p}^u, I \cap T_{w_p}^u)$ -movable. However, this contradicts the assumption that  $w_p$  is  $(\bar{T}, \bar{I})$ -rigid, because  $\bar{T} = T_{w_p}^u$  and  $\bar{I} = I \cap T_{w_p}^u$  in this case.

**Case (2):**  $z \neq u$ . (See Fig. 10(b).)

Let  $w_1$  be the neighbor of  $z$  on the  $vw_p$ -path other than  $w_p$ ; let  $N(T, z) = \{w_1, w_2, \dots, w_p\}$ . We note that the subtree  $T_{w_1}^z$  contains the deleted star  $T \setminus \bar{T}$  centered at  $u$ .



**Fig. 10.** Illustration for Lemma 9.

We first note that the token  $t_p$  on  $w_p$  is  $(\bar{T}_{w_p}^z, \bar{I} \cap \bar{T}_{w_p}^z)$ -rigid, because otherwise  $t_p$  can be slid to some vertex in  $\bar{T}_{w_p}^z$  and hence it is  $(\bar{T}, \bar{I})$ -movable. Since  $\bar{T}_{w_p}^z = T_{w_p}^z$  and  $\bar{I} \cap \bar{T}_{w_p}^z = I \cap T_{w_p}^z$ , the token  $t_p$  is also  $(T_{w_p}^z, I \cap T_{w_p}^z)$ -rigid.

For each  $j \in \{2, 3, \dots, p-1\}$  with  $w_j \in I$ , since  $t_p$  is  $(T_{w_p}^z, I \cap T_{w_p}^z)$ -rigid and all tokens in  $I$  are  $(T, I)$ -movable, by Lemma 2 each token  $t_j$  on  $w_j$  is  $(T_{w_j}^z, I \cap T_{w_j}^z)$ -movable. Then, since  $T_{w_j}^z = \bar{T}_{w_j}^z$  and  $I \cap T_{w_j}^z = \bar{I} \cap \bar{T}_{w_j}^z$ , the token  $t_j$  is  $(\bar{T}_{w_j}^z, \bar{I} \cap \bar{T}_{w_j}^z)$ -movable. Therefore, if  $w_1 \notin \bar{I}$  or the token  $t_1$  on  $w_1$  is  $(\bar{T}_{w_1}^z, \bar{I} \cap \bar{T}_{w_1}^z)$ -movable, then we can slide  $t_p$  from  $w_p$  to  $z$  after sliding each token  $t_j$  in  $\bar{I} \cap \{w_1, w_2, \dots, w_{p-1}\}$  to some vertex of the subtree  $\bar{T}_{w_j}^z$ . This contradicts the assumption that  $t_p$  is  $(\bar{T}, \bar{I})$ -rigid.

Therefore, we have  $w_1 \in \bar{I}$  and a token  $t_1$  on  $w_1$  is  $(\bar{T}_{w_1}^z, \bar{I} \cap \bar{T}_{w_1}^z)$ -rigid. However, since  $t_p$  is  $(\bar{T}_{w_p}^z, \bar{I} \cap \bar{T}_{w_p}^z)$ -rigid, this implies that  $t_1$  is  $(\bar{T}, \bar{I})$ -rigid. Since  $w_1$  is on the  $vw_p$ -path in  $T$ , this contradicts the assumption that  $t_p$  is the  $(\bar{T}, \bar{I})$ -rigid token closest to  $v$ .  $\square$

### Proof of the if-part of Lemma 7.

We now prove the if-part of the lemma by the induction on the number of tokens  $|I_b| = |I_r|$ . The lemma clearly holds for any tree  $T$  if  $|I_b| = |I_r| = 1$ , because  $T$  has only one token and hence we can slide it along the unique path in  $T$ .

We choose an arbitrary safe degree-1 vertex  $v$  of a tree  $T$ , whose unique neighbor is  $u$ . Since all tokens in  $I_b$  are  $(T, I_b)$ -movable, by Lemma 8 we can obtain an independent set  $I'_b$  of  $T$  such that  $v \in I'_b$  and  $I_b \overset{T}{\rightsquigarrow} I'_b$ . By Lemma 9 all tokens in  $I'_b \setminus \{v\}$  are  $(\bar{T}, I'_b \setminus \{v\})$ -movable, where  $\bar{T}$  is the subtree defined in Lemma 9. Similarly, we can obtain an independent set  $I'_r$  of  $T$  such that  $v \in I'_r$ ,  $I_r \overset{T}{\rightsquigarrow} I'_r$  and all tokens in  $I'_r \setminus \{v\}$  are  $(\bar{T}, I'_r \setminus \{v\})$ -movable. Apply the induction hypothesis to the pair of independent sets  $I'_b \setminus \{v\}$  and  $I'_r \setminus \{v\}$  of  $\bar{T}$ . Then, we have  $I'_b \setminus \{v\} \overset{\bar{T}}{\rightsquigarrow} I'_r \setminus \{v\}$ . Recall that both  $u \notin I'_b$  and  $u \notin I'_r$  hold, and  $u$  is the unique neighbor of  $v$  in  $T$ . Furthermore,  $u \notin V(\bar{T})$ . Therefore, we can extend the reconfiguration sequence in  $\bar{T}$  between  $I'_b \setminus \{v\}$  and  $I'_r \setminus \{v\}$  to a reconfiguration sequence in  $T$  between  $I'_b$  and  $I'_r$ . We thus have  $I_b \overset{T}{\rightsquigarrow} I_r$ .

This completes the proof of Lemma 7, and hence completes the proof of Theorem 1.  $\square$

### 3.4 Length of reconfiguration sequence

In this subsection, we show that an actual reconfiguration sequence can be found for a yes-instance on trees, by implementing our proofs in Section 3.2. Furthermore, the length of the obtained reconfiguration sequence is at most quadratic.

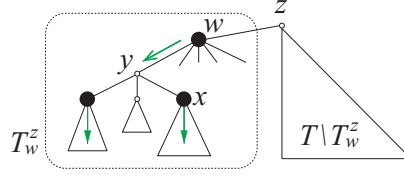
**Theorem 2.** *Let  $I_b$  and  $I_r$  be two independent sets of a tree  $T$  with  $n$  vertices. If  $I_b \overset{T}{\rightsquigarrow} I_r$ , then there exists a reconfiguration sequence of length  $O(n^2)$  between  $I_b$  and  $I_r$ , and it can be output in  $O(n^2)$  time.*

We note that a reconfiguration sequence  $\mathcal{S}$  can be represented by a sequence of edges on which tokens are slid. Therefore, the space for representing  $\mathcal{S}$  can be bounded by a linear in the length of  $\mathcal{S}$ .

By Theorem 1 we can determine whether  $I_b \overset{T}{\rightsquigarrow} I_r$  or not in  $O(n)$  time. In the following, we thus assume that  $I_b \overset{T}{\rightsquigarrow} I_r$ . Furthermore, suppose that all tokens in  $I_b$  are  $(T, I_b)$ -movable, and that all tokens in  $I_r$  are  $(T, I_r)$ -movable; otherwise we obtain the forest by deleting the vertices in  $N[T, \mathcal{R}(I_b)] = N[T, \mathcal{R}(I_r)]$  from  $T$ , and find a reconfiguration sequence for each tree in the forest, according to Lemma 6.

As in the if-part proof of Lemma 7, we choose an arbitrary safe degree-1 vertex  $v$  of  $T$ , and obtain an independent set  $I'_b$  of  $T$  such that  $v \in I'_b$  and  $I_b \overset{T}{\rightsquigarrow} I'_b$ , as follows.

- (a) Find a vertex  $w \in I_b$  which is closest to  $v$ , and let  $P = (v, p_1, p_2, \dots, p_{\ell-1}, w)$  be the  $vw$ -path in  $T$ . Let  $M' = I_b \cap N(T, p_{\ell-1})$ . (See also Fig. 9.)



**Fig. 11.** Illustration for Lemma 10.

- (b) Choose a vertex  $w'$  such that the token on  $w'$  is  $(T_{w'}^{p_\ell-1}, I \cap T_{w'}^{p_\ell-1})$ -rigid if it exists, otherwise choose an arbitrary vertex in  $M'$  and regard it as  $w'$ .
- (c) Slide each token on  $w'' \in M' \setminus \{w'\}$  to some vertex in  $T_{w''}^{p_\ell-1}$ , and then slide the token on  $w'$  to  $v$ .

In Lemma 8 we have proved that such a reconfiguration sequence from  $I_b$  to  $I'_b$  always exists. We apply the same process to  $I_r$ , and repeat until we obtain the same independent set  $I^*$  of  $T$  such that  $I_b \overset{T}{\longleftrightarrow} I^*$  and  $I_r \overset{T}{\longleftrightarrow} I^*$ . Note that, since any reconfiguration sequence is reversible, this means that we obtained a reconfiguration sequence between  $I_b$  and  $I_r$ .

Therefore, to prove Theorem 2, it suffices to show that the algorithm above runs in  $O(n)$  time for one safe degree-1 vertex  $v$  and the reconfiguration sequence for sliding one token to  $v$  is of length  $O(n)$ . In particular, the following lemma completes the proof of Theorem 2.

**Lemma 10.** *Let  $I$  be an independent set of a tree  $T$ , and let  $w \in I$ . For a neighbor  $z \in N(T, w)$ , suppose that the token on  $w$  is  $(T_w^z, I \cap T_w^z)$ -movable. Then, there exists a reconfiguration sequence  $\mathcal{S}_w$  of length  $O(|V(T_w^z)|)$  from  $I$  to an independent set  $I'$  of  $T$  such that  $w \notin I'$  and  $J \cap (T \setminus T_w^z) = I \cap (T \setminus T_w^z)$  for all  $J \in \mathcal{S}_w$ . Furthermore,  $\mathcal{S}_w$  can be output in  $O(|V(T_w^z)|)$  time.*

*Proof.* We prove the lemma by the induction on the depth of  $T_w^z$ , where the depth of a tree is the longest distance from its root to a leaf. If the depth of  $T_w^z$  is zero (and hence  $T_w^z$  consists of a single vertex  $w$ ), then the token on  $w$  is  $(T_w^z, I \cap T_w^z)$ -rigid; this contradicts the assumption. Therefore, we may assume that the depth is at least one. If the depth of  $T_w^z$  is exactly one, then  $T_w^z$  is a star centered at  $w$ , and no token is placed on any neighbor of  $w$ . Thus, we can slide the token on  $w$  by 1 ( $< |V(T_w^z)|$ ) token-slides. Then, the lemma holds for trees with depth one.

Assume that the depth of  $T_w^z$  is  $k \geq 2$ , and that the lemma holds for trees with depth at most  $k-1$ . Since  $w$  is  $(T_w^z, I \cap T_w^z)$ -movable, by Lemma 1 there is a vertex  $y \in N(T_w^z, w)$  such that all tokens on the vertices  $x$  in  $I \cap N(T_y^z, y)$  are  $(T_x^y, I \cap T_x^y)$ -movable. (See Fig. 11.) Then, we can obtain a reconfiguration sequence which (1) first slides all tokens on the vertices  $x$  in  $I \cap N(T_y^z, y)$  to some vertices in  $T_x^y$ , and (2) then slide the token on  $w$  to the vertex  $y$ . By applying the induction hypothesis to each subtree  $T_x^y$ , this reconfiguration sequence is of length

$$1 + \sum_{x \in I \cap N(T_y^z, y)} O(|V(T_x^y)|) = O(|V(T_y^z)|),$$

and can be output in time  $O(|V(T_y^z)|)$ . Note that  $w \notin I'$  holds for the obtained independent set  $I'$  of  $T$ . Thus, the lemma holds for trees with depth  $k$ .  $\square$

It is interesting that there exists an infinite family of instances on paths for which any reconfiguration sequence requires  $\Omega(n^2)$  length, where  $n$  is the number of vertices. For example, consider a path  $(v_1, v_2, \dots, v_{8k})$  with  $n = 8k$  vertices for any positive integer  $k$ , and let  $I_b = \{v_1, v_3, v_5, \dots, v_{2k-1}\}$  and  $I_r = \{v_{6k+2}, v_{6k+4}, \dots, v_{8k}\}$ . In this yes-instance, any token must be slid  $\Theta(n)$  times, and hence any reconfiguration sequence requires  $\Theta(n^2)$  length to slide them all.



**Fig. 12.** No-instance for an interval graph such that all tokens are movable.

## 4 Concluding Remarks

In this paper, we have developed an  $O(n)$ -time algorithm to solve the SLIDING TOKEN problem for trees with  $n$  vertices, based on a simple but non-trivial characterization of rigid tokens. We have shown that there exists a reconfiguration sequence of length  $O(n^2)$  for any yes-instance on trees, and it can be output in  $O(n^2)$  time. Furthermore, there exists an infinite family of instances on paths for which any reconfiguration sequence requires  $\Omega(n^2)$  length.

The complexity status of SLIDING TOKEN remains open for chordal graphs and interval graphs. Interestingly, these graphs have no-instances such that all tokens are movable. (See Fig. 12 for example.)

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