The Life of Quanta: Entanglement, Wormholes and the Second Law

by

Hrant Gharibyan

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Abstract

This thesis explores two different topics in physics.

The first is related to the study of the ER = EPR conjecture that relates the entanglement entropy of a collection of black holes to the cross sectional area of Einstein-Rosen (ER) bridges (or wormholes) connecting them. We show that the geometrical entropy of classical ER bridges satisfies the subadditivity, triangle, strong subadditivity, and CLW inequalities. These are nontrivial properties of entanglement entropy, so this is evidence for ER = EPR. We further show that the entanglement entropy associated to classical ER bridges has nonpositive tripartite information. This is not a property of entanglement entropy, in general. For example, the entangled four qubit pure state $|GHZ_4\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}$ has positive tripartite information, so this state cannot be described by a classical ER bridge. Large black holes with massive amounts of entanglement between them can fail to have a classical ER bridge if they are built out of $|GHZ_4\rangle$ states. States with nonpositive tripartite information are called monogamous. We conclude that classical ER bridges require monogamous EPR correlations.

The second is a generalization of the second law of thermodynamics. We prove a generalization of the classic Groenewold-Lindblad entropy inequality, combining decoherence and the quantum Bayes theorem into a simple unified picture where decoherence increases entropy while observation decreases it. This provides a rigorous quantum-mechanical version of the second law of thermodynamics, governing how the entropy of a system evolves under general decoherence and observation. The powerful tool of spectral majorization enables both simple alternative proofs of the classic Lindblad and Holevo inequalities without using strong subadditivity, and also novel inequalities for decoherence and observation that hold not only for von Neumann entropy, but also for arbitrary concave entropies.

Thesis Supervisor: Max Tegmark
Title: Professor of Physics

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This thesis is dedicated to the memory of my grandfathers, Albert Avagyan and Haykaz Gharibyan, who took me under their wing as a little kid and taught me THE IMPORTANT. I would like to thank my parents, Manush and Sargis, for giving me life and giving me love. Thanks for being next to me, no matter what. Thanks to my brother, Haykaz, for being a true friend and life companion. Thanks to my high school teacher, Gagik Vardanich, for introducing me to the wonders and pleasures of physics. Thanks to Bob Penna for numerous conversations about black holes, wormholes and entanglement. Thanks to my friend Arvin for nightlong discussions on information paradox and holography. Thanks to my undergraduate advisor Edward Farhi for continuous support and guidance. Thanks to Seth Lloyd for being the coolest quantum mechanical engineer and Peter Shor for teaching me the best of quantum information theory. Thanks to all of my friends: Wojciech, Rudy, Grace, Leo, Katelin, Jeff, Justin, Ioana, Charles and Illan. You guys taught me a lot! Finally, thank you Max for being the most AWESOME mentor in our Galaxy. Your curiosity about nature, your passion for science and mentorship have inspired me throughout my time at MIT. Thank you for an amazing four years of learning, creating, discovery and friendship. It has been a lot of fun!

Preface

This thesis is a discussion around two different topics I worked on as an undergraduate student at MIT. The results presented here have been previously published in scientific articles, but the goal of this thesis is to present them in a unified picture exploiting their unexpected relation to quantum information theory. Quantum information theory brings together two of the greatest scientific ideas of the twentieth century: the quantum mechanics and information theory. Quantum mechanics was established by 1930 and Shannon's information theory by 1948, but it took more then 30 years for those ideas to converge. The formulation of quantum information theory is sometimes referred to second quantum revolution [1], the first being the discovery of quantum mechanics. Quantum information theory revolutionized our understanding of computation and communication, significantly pushing the boundaries of what is computation. Entirely new direction of research was established, aiming to build quantum computer, develop algorithms for quantum computation and protocols for quantum communication. Very recently, it was also recognized that numerous questions in physics can be addressed using mathematical toolkit developed by quantum information theorists. Those questions span almost all areas of physics research; from many body systems in condensed matter to gravity and unitary cosmology. As an undergraduate student at MIT, I was given a unique opportunity to learn quantum information theory from the founders and giants: Peter Shor, Seth Lloyd and Edward Farhi. And have had numerous discussions with physicists whose main research is in another sub-field, but who are fascinated by the possible applications of quantum information theory to all of physics: Max Tegmark, Robert Penna and Xiao-Gang Wen. These discussions greatly influenced my interests in physics and information theory. As a result, I greatly enjoyed working on two projects in theoretical physics that use various ideas from quantum information to study physics problems. The first project uses entropic inequalities from quantum information theory to investigate the unexpected link between geometrical wormholes and quantum entanglement. The second, uses information inequalities and majorization techniques to generalize the second law of thermodynamics. The thesis is structured as follows:

In Chapter 1, we introduce the quantum theory of microscopic objects and formulate Shannon information theory for quantum systems. We further introduce various measures to understand and quantify entropy, information, and correlations in quantum systems. Then, we discuss important information inequalities that these measures obey and information theoretical properties that quantum systems have. This section is meant to be a quick review of the subject and might not be detailed enough for somebody who have never seen this material before.

In Chapter 2, we review basics of classical black hole physics and black hole thermodynamics. We use semiclassical approach to investigate quantum properties of black holes and derive Hawking radiation. This derivation suggest that quantum black holes evaporate from pure states into mixed states, contradicting to the fact that quantum information can not be erased. This contradiction is known as information paradox. We further discuss several proposals that use arguments from quantum information theory to address this paradox. The purpose of this chapter is to provide review on black hole physics and information paradox, and to motivate the discussion of next chapter, that will focus on a particular proposal which indirectly addresses this paradox.

In Chapter 3, we motivate and present the ER = EPR conjecture that proposes a duality between geometrical wormholes and quantum entanglement. We present the analogous study of this duality in AdS/CFT framework and discuss it's generalization to the case of entangled black holes in an asymptotically flat space. The ER = EPR conjecture relates the entanglement entropy of a collection of black holes to the cross geometrical area of the wormhole connecting them. We provide evidence for this conjecture by showing that geometrical entropy of classical ER bridges satisfy the nontrivial properties of entanglement entropy. We further show that the entanglement entropy associated with classical ER bridges has non-positive interaction information. This is not a general property of entanglement entropy and thus provides an additional constraints on the conjecture. Lastly, we conclude by describing several advances and open problems in the study of the link between geometry and quantum information theory.

In Chapter 4, we deploy the assumption of the unitary cosmology to generalize the second law of thermodynamics. We prove a generalization of the classic Groenewold-Lindblad entropy inequality, combining decoherence and the quantum Bayes theorem into a simple unified picture where decoherence increases entropy while observation decreases it. The powerful tool of spectral majorization enables both simple alternative proofs of the classic Lindblad and Holevo inequalities without using strong subadditivity, and also novel inequalities for decoherence and observation. Lastly, we discuss how this generalization helps to understand and interpret the entropy problem of inflationary universe.

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Chapter 1

Quantum Information Theory

1.1 Quantum Theory

Quantum revolution of physics started by Planck's proposal that light comes in discrete bundles. A few years later, in 1905, Einstein published a paper that used Planck's proposal to demonstrate that the postulate that light arrives in quanta provides a simple explanation for the photoelectric effect. Then, Louis de Broglie postulated that every individual element of matter, whether an atom, electron, or photon, has both particle-like behavior and wave-like behavior. These principles were generalized by Schrdinger, Heisenberg and Dirac into a mathematical theory of quantum mechanics, which describes the world of microscopic object with high accuracy. The standard formulation of quantum mechanics is based on four fundamental axioms, which serve as a basis for describing isolated quantum systems.

1.1.1 Fundamental Concepts

In this section, we will not present those axioms, but will instead talk about several important properties that quantum systems possess. Some this properties are uniquely quantum and have no classical analog.

- **Superposition** principle states that a quantum system can be in a linear combination state of any two other states, which is due to the linearity of quantum theory. If a particle can be found in two different points in space, then it can be in any linear superposition of those points. One can interpret it as a strange property of quantum particles that can be in two places at the same time.

- The quantum theory is **indeterministic**, as it only predicts probabilities of possible outcomes. Two continue with the previous example, assuming the particle is in a superposition state of two points in space, then if we attempt to measure it's positions we will find it only in one of the locations. The outcome of the measurement will not be fully determined, but rather each outcome will be assigned a probability.

- **Uncertainty** is at the heart of the quantum theory. Quantum mechanical uncertainty is fundamentally different from any other classical uncertainty you have ever encountered. In quantum theory, if we measure precisely the position of a particle, then we will not know anything about it's momentum and vice verse. This can be thought as a mixture of superposition and measurement in quantum theory, where any accurate measurement of one property can disturb the superposition of the state in some other property.

- **Entanglement** is the most astonishing property of quantum theory. Schrdinger used the term entanglement to describe the strong quantum correlations between composite quantum systems. It has been later noticed by many prominent scientists, who refused to believe in it. For instance, Einstein, Podolsky, and Rosen thought that such correlations are due to the incompleteness of the quantum theory, and should be possible to explain with classical analogs. Years later John Bell came up with a method to prove that entanglement has no explanation in terms of classical correlations but is instead a uniquely quantum phenomenon. As we will see later in this thesis, entanglement not only plays a fundamental role in quantum information science, but also it is important for understanding relationship between quantum physics and general relativity.

Those properties encompass the most important aspects of quantum theory and serve as a good basis for our journey in the world of quantum information.

1.1.2 Quantum State and Unitary Dynamics

In previous section we introduced quantum theory and described the most important properties that distinguish it from any other classical theory. Here we will develop a mathematical framework to describe the simplest quantum systems and relate it to information processing. In quantum mechanics, classically continuous variables such as energy, angular momentum and charge come in discrete units called quanta. This discrete character of quantum systems such as photons, atoms, and electrons allows them to register ordinary digital information. For instance, a left polarized photon can register 0, whereas right polarized will be 1. In addition, due to superposition principle quantum systems can also register information that classical systems cannot. For instance, photon can be in a superposition of left and right polarized states, thus registering both 0 and 1 simultaneously. To put this into rigorous mathematical framework we start with the simplest quantum system qubit. **Qubit** is a two state quantum system. We use $|0\rangle$ to denote one possible state of the system and $|1\rangle$ to denote the other. We can encode a classical bit or cbit into a qubit with the following mapping:

$$0 \rightarrow |0\rangle \quad \text{and} \quad 1 \rightarrow |1\rangle \tag{1.1.1}$$

This notation was developed by Paul Dirac and was named "ket", which is widely used for calculations in quantum mechanics. The superposition principle of quantum mechanics predicts that any linear combination of those states are also possible states that quantum system can be in. Thus the general *superposition* "ket" state of the qubit is given by

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{1.1.2}$$

where the coefficient are arbitrary complex numbers with unit norm:

$$|\alpha|^2 + |\beta|^2 = 1 \tag{1.1.3}$$

The coefficients α and β are called probability amplitudes, as square of their absolute values correspond to different outcomes of the measurement. To describe the state $|0\rangle$

and $|1\rangle$ the Boolean algebra is not sufficient and the simplest mathematical structure that can contain superposition property is linear algebra. Thus, we can think of this elements as vectors in 2D space

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{1.1.4}$$

With this convention thus the general state of a qubit is given by a vector

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{1.1.5}$$

Where α and β are arbitrary coefficients such that $|\alpha|^2 + |\beta|^2 = 1$. In linear algebra row vectors are also important. The Dirac notation has an entity named a "bra", that can be represented as row vector. So, that the "bra" corresponding to vector $|\psi\rangle$ is

$$\langle\psi| = (|\psi\rangle)^\dagger = [\alpha^*, \beta^*] \tag{1.1.6}$$

where dagger ($\dagger$) denotes operation of matrix transpose followed by complex conjugation. In a vector field language "bra" represents the dual vector of the state vector. For the state $|\psi\rangle$ to be physical, it has to be normalizable, which in a bra-ket notation can be written as:

$$\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 = 1 \tag{1.1.7}$$

This mathematical formalism for qubit can be generalized to systems with arbitrary number of states. Thus, the state of general quantum system will be given by normalized vector $|\psi\rangle$ in a mathematical vector space $\mathcal{H}$, called Hilbert space. In general, the evolution of a closed quantum system will be given by unitary operator $\mathbf{U}$. Unitary evolution in quantum theory is reversible, as a unitary operator always possesses an inverse, which is denoted by $\mathbf{U}^\dagger$. This property gives the relations:

$$\mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U} = \mathbf{I} \tag{1.1.8}$$

The unitary property also ensures that evolution preserves the unit-norm of the state, thus preserving the physicality of the final state $\mathbf{U}|\psi\rangle$:

$$(\mathbf{U}|\psi\rangle)^{\dagger}\mathbf{U}|\psi\rangle = \langle\psi|\mathbf{U}^{\dagger}\mathbf{U}|\psi\rangle = \langle\psi|\psi\rangle = 1 \tag{1.1.9}$$

1.1.3 Measurement

In previous section, we showed that the state of quantum system is given by a Dirac vector $|\psi\rangle \in \mathcal{H}$ and the time evolution of closed quantum system is governed by unitary operations. We also noted that unitary evolution is reversible and preserves the unit norm. Here we will talk about the measurement process in quantum systems and it's fundamental difference from classical systems. As discussed in the previous section, if we have a qubit in superposition state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{1.1.10}$$

then if we try to measure it to be $|0\rangle, |1\rangle$ then the measurement postulate of the quantum theory, also known as the *Born rule*, states that the system we will found in state $|0\rangle$ with probability $p_0 = |\alpha|^2$ and in state $|1\rangle$ with probability $p_1 = |\beta|^2$. This probabilistic mapping of wavefunction from superposition state to basis state is referred to as "collapse". In general setting, various parameters of the system can be measured such as the spin, positions and momentum. For each measurement, the superposition state will "collapse" into the basis state of the observable that is measured. For the general quantum system A in a state $|\psi\rangle \in \mathcal{H}_A$ and some basis of the Hilbert space $|a_i\rangle$, we denote by $\mathbf{P}_i = |a_i\rangle\langle a_i|$ projection into basis vector $|a_i\rangle$. We say that the basis $|a_i\rangle$ is orthonormal and complete if

$$\begin{aligned} \mathbf{P}_i\mathbf{P}_j &= \delta_{ij}\mathbf{P}_i \\ \sum_i \mathbf{P}_i &= \mathbf{I} \end{aligned} \tag{1.1.11}$$

Then, measuring the observable related to selected basis we will find the system in state

$$|\psi_i\rangle = \frac{\mathbf{P}_i|\psi\rangle}{\sqrt{\langle\psi|\mathbf{P}_i|\psi\rangle}} \tag{1.1.12}$$

with probability $p(i) = \langle\psi|\mathbf{P}_i|\psi\rangle$.

1.1.4 Composite Systems and Entanglement

A single physical qubit is the simplest system that exhibits quantum properties, but it is not sufficient for performing useful computational tasks. In order to address this issue we will discuss physical and mathematical properties of composite systems, when we have more then one qubit. Let's start with the simple case of two qubit systems. Analogous to single qubit case we can represent the classical two cbit state as Dirac vector

$$00 \rightarrow |0\rangle|0\rangle \quad \text{or} \quad |00\rangle \tag{1.1.13}$$

Doing this for all cbit combinations and using the superposition principle of quantum theory, we can conclude that the general state of two system will be given by

$$|\xi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \tag{1.1.14}$$

The unit-norm condition $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$ should still hold true for the two-qubit state to be physical. It turns out that composite quantum systems also has a nice representation in linear algebra, where the tensor product operation serves as powerful tool for writing down the state of composite systems. Recalling from linear algebra, tensor product is defined as

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \otimes \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 \\ a_1b_2 \\ b_1a_2 \\ b_1b_2 \end{bmatrix} \tag{1.1.15}$$

Using the definition of tensor product and vector representation of single qubit from previous section, we indeed observe that two-qubit basis states have the following vector representation in 4-dimensional composite Hilbert space:

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{1.1.16}$$

and general two-qubit state can be written as

$$|\xi\rangle = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} \tag{1.1.17}$$

The dynamics of the closed composite systems is also given by unitary operators, which act on the state of composite system. The most interesting unitary operators are the ones, which are impossible to decompose into tensor product of unitary operators acting on individual qubits. This property is widely used in quantum information theory to introduce interactions between qubits, that serve to the computational power of the system. For instance, controlled-U gate uses the state of the first qubit to control the unitary operation $\mathbf{U}$ on the second qubit. Another important thing to discuss is the measurement process in composite systems. As discussed in Section 1.1.3, the measurement on a closed quantum system can be thought as projection into basis states with given probabilities. For the two qubit state $|\xi\rangle$ in Equation (1.1.14) in order to perform measurement on both qubits we will use the projectors:

$$\mathbf{P}_{00} = |00\rangle\langle 00|, \quad \mathbf{P}_{01} = |01\rangle\langle 01|, \quad \mathbf{P}_{10} = |10\rangle\langle 10|, \quad \mathbf{P}_{11} = |11\rangle\langle 11| \tag{1.1.18}$$

Then, according to Born's rule the probabilities of each of the outcomes will be given by:

$$|\alpha|^2 = \langle\xi|\mathbf{P}_{00}|\xi\rangle, \quad |\beta|^2 = \langle\xi|\mathbf{P}_{01}|\xi\rangle, \quad |\gamma|^2 = \langle\xi|\mathbf{P}_{10}|\xi\rangle, \quad |\delta|^2 = \langle\xi|\mathbf{P}_{11}|\xi\rangle \tag{1.1.19}$$

But now we want to know what would happen if we measure only one of the qubits. What are possible final states and what are the probabilities? The answer turns out to be simple. To get the correct prediction, we apply projection operators on the first qubit and the identity operator to the second qubit, as no measurement occurs on it. So, we apply the set of measurement operators:

$$\{\mathbf{P}_0 \otimes \mathbf{I}, \mathbf{P}_1 \otimes \mathbf{I}\} \tag{1.1.20}$$

Thus, after the measurement of the first qubit the final normalized state will be:

$$\frac{\mathbf{P}_0 \otimes \mathbf{I}|\xi\rangle}{\sqrt{\mathbf{P}_0 \otimes \mathbf{I}|\xi\rangle}} = \frac{\alpha|00\rangle + \beta|01\rangle}{\sqrt{|\alpha|^2 + |\beta|^2}} \tag{1.1.21}$$

with probability $\langle\xi|\mathbf{P}_0 \otimes \mathbf{I}|\xi\rangle = |\alpha|^2 + |\beta|^2$ or it will be:

$$\frac{\mathbf{P}_1 \otimes \mathbf{I}|\xi\rangle}{\sqrt{\mathbf{P}_1 \otimes \mathbf{I}|\xi\rangle}} = \frac{\alpha|10\rangle + \beta|11\rangle}{\sqrt{|\gamma|^2 + |\delta|^2}} \tag{1.1.22}$$

with probability $\langle\xi|\mathbf{P}_1 \otimes \mathbf{I}|\xi\rangle = |\gamma|^2 + |\delta|^2$. Now, we are finally in a good position to observe and understand one of the most unique features that composite quantum systems exhibit: **entanglement**. Suppose two physicist, Alice and Bob, each have one qubit and they travel with their qubits to the opposite ends of our universe, many many lightyears away from each other. If our qubits were initially in a state:

$$|00\rangle = |0\rangle^A|0\rangle^B \tag{1.1.23}$$

then they would each know that their own qubit is in a state $|0\rangle$ independent of others. But now imagine if the qubits were created in a special state

$$|\Phi\rangle^{AB} = \frac{|0\rangle^A|0\rangle^B + |1\rangle^A|1\rangle^B}{\sqrt{2}} \tag{1.1.24}$$

Alice and Bob still each possess one specific qubit, but they can't really tell what is the exact quantum state of their own qubit. It seems that they can only say what is the state of composite system, but not individual qubit. This is very strange connection between this two qubits, who are now very far from each other. Furthermore, suppose Alice measures the first qubit and finds it in state $|0\rangle^A$, then according to projection rule from Equation (1.1.21) the two qubit state will "collapse" into $|0\rangle^A|0\rangle^B$. Similarly, if Alice measures $|1\rangle^A$, the total state will be $|1\rangle^A|1\rangle^B$ and Bob's qubit will be in state $|1\rangle^B$. This means that Bob's qubit immediately learns about the outcome of Alice's measurement and picks it's own state appropriately. It doesn't matter how far Bob and Alice are, the effect is instantaneous. To make it clear, entanglement is special type of correlation between quantum systems that have been created together, but it does not provide means for sending superluminar signals and doesn't violate principle of relativity. Entanglement is an important resource for quantum computation and quantum communication. For instance, superdense coding by Bennett and Wiesner [2] is a technique used to send two bits of classical information using only one qubit, with the aid of entanglement. Entanglement assisted communication is an active area of research in information theory.

1.1.5 No Cloning Theorem

The *no cloning theorem* [3] states that it is impossible to build a device that can copy arbitrary quantum state. It is a simplest result in the quantum theory, yet it has very profound consequences on quantum information processing. It may seem rather strange that quantum information can not be copied, as coping classical information is an easy task. To prove this theorem, we suppose that there exists a copying device $\mathbf{U}$ that takes two qubits and copies arbitrary state of the first into second qubit.

Consider two arbitrary states of the first qubit $|\psi\rangle$ and $|\phi\rangle$, which are copied into second qubit:

$$\begin{aligned} \mathbf{U}|\psi\rangle|0\rangle &= |\psi\rangle|\psi\rangle \\ \mathbf{U}|\phi\rangle|0\rangle &= |\phi\rangle|\phi\rangle \end{aligned} \tag{1.1.25}$$

Then, taking the dot product of this two-qubit states we get:

$$\langle\psi|\langle\psi||\phi\rangle|\phi\rangle = \langle\psi|\langle 0|\mathbf{U}^{\dagger}\mathbf{U}|\phi\rangle|0\rangle \tag{1.1.26}$$

Using that $\mathbf{U}^{\dagger}\mathbf{U} = \mathbf{I}$ and $\langle 0|0\rangle = 1$ we obtain:

$$\langle\psi|\langle\psi||\phi\rangle|\phi\rangle = \langle\psi|\phi\rangle^2 = \langle\psi|\phi\rangle \tag{1.1.27}$$

This relation is true only for $\langle\psi|\phi\rangle = 0$ and $\langle\psi|\phi\rangle = 1$, meaning $|\phi\rangle$ and $|\psi\rangle$ have two be either the same or orthogonal. Thus, it is impossible to copy quantum information for arbitrary states contradicting our assumption that such a universal copying device exists. Analogously one can show that quantum information can not be deleted using unitary operations. This leads to the conclusion that in a quantum world information is never created and never lost, which is rooted in the fact that closed quantum systems undergo unitary evolution. We will discuss this in more detail and with mathematical formalism in following sections.

1.1.6 Density Matrix Formalism

The standard state vector formulation of quantum mechanics given above is a very convenient tool for describing systems whose quantum state is completely known. In a more realistic situations the state of the system might not be fully known, but rather will be described by some statistical ensemble $\{p_i, |\psi_i\rangle\}$, where the system is in one of the states $|\psi_i\rangle$ with given probability p_i. In this case we say that system is in the *pure state* if the ensemble $\{p_i, |\psi_i\rangle\}$ consists of single element with probability one and we say system in a *mixed state* if it has at least two elements with non-zero probabilities. In order to provide convenient means for describing such systems we define so called density operator or density matrix ρ as:

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| . \tag{1.1.28}$$

The class of these operators has the following properties

1. ρ is self-adjoint: $\rho = \rho^\dagger$
2. ρ has trace equal to one ($\mathrm{tr}\,(\rho) = 1$)
3. ρ is a positive operator (for an arbitrary state $|\psi\rangle$, $\langle\psi|\rho|\psi\rangle \geq 0$

This properties follow from the definition of the density matrix and the fact that $\sum_i p_i = 1$. Density matrix formulation of quantum mechanics provides powerful means for studying statistical properties of quantum systems, which is relevant for noisy communication. Suppose we have a closed quantum system described by the ensemble of pure states $\{p_i, |\psi_i\rangle\}$, with an associated density operator ρ given by Equation (1.1.28). Following are the standard set of axioms of quantum mechanics in the new language of density operators.

State: The system is completely described by its density operator, which is a positive operator ρ with trace one, acting on the state space of the system.

Dynamics: Time evolution of the density operator of the isolated quantum system is given by the formula:

$$\rho \to \sum_i p_i U \left|\psi_i\right\rangle \left\langle\psi_i\right| U^\dagger = U\rho U^\dagger \tag{1.1.29}$$

Measurement: If system is measurement by projection operators $\{\mathbf{P}_i\}$, the post-measurement density operator is

$$\rho_i = \frac{\mathbf{P}_i \rho \mathbf{P}_i}{\mathrm{tr}\,(\mathbf{P}_i \rho)} \tag{1.1.30}$$

with the probability $p(i) = \mathrm{tr}\,(\mathbf{P}_i \rho)$.

Composite System: The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. $(\rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_n)$

This axioms together provide complete mathematical framework for doing information theory in quantum systems.

1.1.7 Reduced Density Operator

Suppose we have a physical system composed of subsystems A and B, whose total state is described by the density operator ρ^{AB}. What is the density matrix of subsystem A? In section Section 1.1.4 we saw that Alice had trouble describing the state of her qubit only, when two qubits were in an entangled state given by Equation (1.1.24). To address this problem, we define reduced density operator for subsystem A as:

$$\rho^A = \mathrm{tr}_B(\rho^{AB}) \tag{1.1.31}$$

where tr_B is the partial trace over subsystem B. The partial trace operation is defined:

$$\mathrm{tr}_B(\rho^{AB}) = \sum_{i=1}^{N} \langle b_i | \rho^{AB} | b_i \rangle \tag{1.1.32}$$

where $|b_i\rangle$ is an arbitrary orthonormal basis of the subspace B. The reduction method is physically justified because it provides correct measurement statistics. A useful example is the case of non-entangled qubit, where the system is in a product state $\rho^{AB} = \rho^A \otimes \rho^B$. Then the reduced density operator is given by

$$\mathrm{tr}_B(\rho^A \otimes \rho^B) = \rho^A \mathrm{tr}\,(\rho^B) = \rho^A \tag{1.1.33}$$

which is the result we intuitively expected. Now let's look into the case of entangled subsystems. For the simple example, when A and B are the qubits that Alice and Bob possess in an entangled state $\rho^{AB} = |\Phi\rangle\langle\Phi|^{AB}$ given by Equation (1.1.24), the density matrix of the Alice's qubit will be

$$\rho^A = \sum_{j=0}^{1} \langle j|^B \rho^{AB} |j\rangle^B = \frac{1}{2} |0\rangle \langle 0|^A + \frac{1}{2} |1\rangle \langle 1|^A \tag{1.1.34}$$

Thus, we see that reduced density matrix for Alice's qubit is the statistical mixture of two pure state $|0\rangle$ and $|1\rangle$ without any superposition components. This gives an intuitive picture, that if Alice does not have access to Bob's qubit and has no idea what is Bob doing with it. Then the best description of her own qubit would be statistical density matrix that ignores all of the degree of freedom of second qubit (trace over B). This method of analysis serves quantum information theorist in many ways, in particular, when they try to model the interaction of the open quantum system with an environmental or try to find an estimate for the entanglement between two systems.

1.1.8 Purification

We can think of purification as the opposite of tracing over. Suppose we are given a mixed density operator ρ^A on a system A, which has a spectral decomposition of following form:

$$\rho^A = \sum_k p_k |k\rangle \langle k| \tag{1.1.35}$$

where $\{|k\rangle\}$ are eigenvectors of the density matrix ρ^A. Then, we can find a reference system R and pure bipartite state $|\psi\rangle^{RA}$ for composite system AR, such the the reduced density matrix of A is exactly ρ^A:

$$\rho^A = \mathrm{tr}_R(|\psi\rangle \langle \psi|^{RA}) \tag{1.1.36}$$

It turns out that any state ρ^A can be purified and there are many possible ways to do it. The simplest ways is to choose the reference system R to be exact copy of system

A and construct the composite state $|\psi\rangle^{RA}$ to be:

$$|\psi\rangle^{RA} = \sum_k \sqrt{p_k}|k\rangle^R|k\rangle^A \tag{1.1.37}$$

It is quite clear that the reduced density matrix of this state will be initial ρ^A state. Purification provides powerful means for interpreting statistical noise of system A as an entanglement with some system R, that we have no access to. This has no classical analog, as entanglement itself has no classical analog.

1.2 Entropy and Information

1.2.1 von Neumann Entropy

The Shannon entropy measures the uncertainty associated with a classical probability distribution. Quantum states are described in a similar fashion, with density operators replacing probability distributions. In this section we generalize the definition of the Shannon entropy to quantum states. Von Neumann defined [4] the entropy of a quantum state by the formula:

$$S(\rho) = -\mathrm{tr}\,(\rho \log_2 \rho) \tag{1.2.38}$$

According to this definition the entropy of the quantum system is zero if we know the exact pure state of the system and is positive if system is in a mixed state. This definition is independent of the observation process and number of possible outcomes. If λ_i are the eigenvalues of ρ then von Neumann's definition can be re-expressed:

$$S(\rho) = -\sum_i \lambda_i \log \lambda_i \tag{1.2.39}$$

For calculations it is usually this last formula which is most useful. For instance, the completely mixed density operator in a d-dimensional space, I/d, has entropy $\log d$. Several important mathematical properties of von Neumann entropy are:

1. **Positivity:** The von Neumann entropy $S(\rho)$ is non-negative for any density operator ρ: $S(\rho) \geq 0$. This follows from the fact that all eigenvalues of ρ are positive and less then one.

2. **Minimum Value:** The minimum value of the von Neumann entropy is zero, and it occurs when the density operator is a pure state. Analogous to case of Shannon entropy, because $0 \leq \lambda_i \leq 1$ and $\sum_i \lambda = 1$ the minimum value zero is reached when only one of the eigenvalues is 1 and the rest are zero.

3. **Maximum Value:** The maximum value of the von Neumann entropy is log D where D is the dimension of the system, and it occurs for the maximally mixed state. This case is also analogous to classical Shannon entropy, when the maximum is achieved when all the probabilities (eigenvalues) are equal to each other.

4. **Concavity:** The entropy is concave in the density operator:

$$S(\rho) \geq \sum_i p_i S(\rho_i) \tag{1.2.40}$$

where $\rho = \sum_i p_i \rho_i$. We will not provide the proof of this property, but there exist extensive literature that contains the proof to this property.

5. **Unitary Invariance:** The entropy of a density operator is invariant under unitary operations on it: $S(\rho) = S(\mathbf{U}\rho\mathbf{U}^\dagger)$. This is true from the linear algebra point of view, when it is well known fact that the eigenvalues of the operator remain invariant under unitary transformation.

1.2.2 Quantum Relative Entropy

As for the Shannon entropy, it is extremely useful to define a quantum version of the relative entropy. Suppose ρ and σ are density operators. The *relative entropy* of ρ to σ is defined by

$$D(\rho||\sigma) = \mathrm{tr}\,(\rho \log \rho) - \mathrm{tr}\,(\rho \log \sigma) \tag{1.2.41}$$

Similar to the classical case, we can intuitively think of it as a distance measure between two quantum states. In order to proceed in the discussion of various measures of quantum information, one has to know two important properties of the relative entropy:

Klein's Inequality: The relative entropy $D(\rho||\sigma)$ is positive for any two density operators ρ and σ:

$$D(\rho||\sigma) \geq 0 \tag{1.2.42}$$

This is not very surprising result, as the the classical analog exits, but it is not completely the same. The proof of this inequality is slightly more challenging then the classical one, but interested reader can find it in [5] and [6].

Monotonicity: The quantum relative entropy between two states ρ^{AB} and σ^{AB} can only decrease if trace out the system B:

$$D(\rho^{AB}||\sigma^{AB}) \geq D(\rho^{A}||\sigma^{A}) \tag{1.2.43}$$

The proof of this inequality is quite challenging and will not provide it in this thesis, but interested reader can find it in the Appendix of [6]. As we will see later, this inequality is a key step in proving one of the most important properties of von Neumann entropy called strong subadditivity.

1.2.3 Conditional Entropy and Coherent Information

The difference between quantum and classical information becomes evident when we define the conditional quantum entropy. This measure of entropy establishes a clear distinction between quantum and classical correlations and provides means of quantifying them. The **conditional quantum entropy** $S(A|B)$ of a bipartite quantum state ρ^{AB} is the difference of the joint quantum entropy of system AB and marginal quantum entropy of subsystem B.

$$S(A|B) = S(AB) - S(B) \tag{1.2.44}$$

where marginal entropy is calculated for the reduced density matrix of subsystem $\rho^B = \mathrm{tr}_A(\rho^{AB})$. This definition is the natural generalization of the classical conditional entropy and obeys many relations that the classical conditional entropy obeys, such as chaining rules and that conditioning reduces entropy. An obvious difference with classical case is that conditional quantum entropy can take on negative values. For instance, if composite system AB is in a entangled pure state $|\Phi\rangle^{AB}$:

$$|\Phi\rangle^{AB} = \frac{|0\rangle^A|0\rangle^B + |1\rangle^A|1\rangle^B}{\sqrt{2}} \tag{1.2.45}$$

Then $S(AB) = 0$ and we can calculate the $S(B)$ by tracing out the system A:

$$\rho^B = \sum_{j=0}^{1} \langle j|^A \rho^{AB} |j\rangle^A = \frac{1}{2}\,|0\rangle\,\langle 0|^B + \frac{1}{2}\,|1\rangle\,\langle 1|^B \tag{1.2.46}$$

Thus, the entropy $S(B) = 1$ bit, so see that $S(A|B) = -1$bit. Negative of the conditional quantum entropy is an important measure in quantum information theory, we even have an information quantity and a special notation to denote the negative of the conditional quantum entropy. The **coherent information** $I(A\}B)$ of a bipartite state ρ^{AB} is defined as:

$$I(A\}B) = S(B) - S(AB) \tag{1.2.47}$$

The coherent information is an information quantity that measures merely quantum correlations, much like the mutual information does in the classical case. We indeed observe that for the two qubits in an entangled pure state Equation (1.2.45) we have $I(A\}B) = 1$ bit, which indicates that there is about 1 bit of entanglement between this qubits. In general, coherent information is a good measure of entanglement for bipartite systems as it eliminates classical correlations, leaving only the quantum ones.

1.2.4 Entanglement Entropy

Independent of information theorists, physicists have defined another measure to quantify entanglement between systems in condensed matter systems and quantum field theories. They called it **entanglement entropy**, which is a less general case of the coherent information. Suppose the total system is composed of subsystems A and B and total system is in a pure state, then the amount of entanglement that system B has with the rest of the system is given by:

$$S_B = -\mathrm{tr}\, \rho^B \log(\rho^B) \tag{1.2.48}$$

where ρ^B is the reduced density matrix. We observe that $S_B = I(A\}B)$, because system AB is in a pure state (S(AB)=0). The rest of the thesis will be concerned with systems that are in a global pure state and we will refer to entanglement entropy as a measure of quantum correlations.

1.2.5 Quantum Mutual Information: Subadditivity

The standard informational measure of correlations in the classical world is the mutual information, and such a quantity plays a prominent role in quantifying both classical and quantum correlations in the quantum world as well. The **quantum mutual information** between subsystems A and B of a bipartite state ρ^{AB} is given by:

$$I(A:B) = S(A) + S(B) - S(AB) \tag{1.2.49}$$

The following relations hold for quantum mutual information, in analogy with the classical case:

$$\begin{aligned} I(A:B) &= S(A) - S(A|B) \\ &= S(B) - S(B|A) \end{aligned} \tag{1.2.50}$$

which can also be written in terms of **coherent information**:

$$
\begin{aligned}
I(A:B) &= S(A) + I(A\}B) \\
&= S(B) + I(B\}A)
\end{aligned}
\tag{1.2.51}
$$

One of the most important properties of quantum mutual information is that it is positive quantity for any quantum state ρ and subsystems A and B:

$$
I(A:B) \geq 0 \tag{1.2.52}
$$

This property can also be written as:

$$
S(A) + S(B) \geq S(AB) \tag{1.2.53}
$$

which is known as **subadditivity** of von Neumann entropy and is one of the most relevant entropy inequalities in quantum information theory.

1.2.6 Conditional Mutual Information: Strong Subadditivity

The conditional quantum mutual information $I(A;B|C)$ of any tripartite state ρ^{ABC} is similarly to the classical definition:

$$
I(A:B|C) = S(A|C) + S(B|C) - S(AB|C) \tag{1.2.54}
$$

This measure quantifies the classical and quantum correlations between subsystems A and B, when the state of the system C is known. Using definition of the mutual information from previous sections, we can also express condition quantum mutual information as follows:

$$
I(A:B|C) = I(A:BC) - I(A:B) \tag{1.2.55}
$$

In the classical world, conditional mutual information is a positive measure, which is quite easy to see from it's definition and the positivity of mutual information. It is much harder to prove that this quantity is positive for quantum systems:

$$I(A:B|C) \geq 0 \tag{1.2.56}$$

The proof follows from the observation that:

$$\begin{aligned} I(A:BC) &= D(\rho^{ABC}||\rho^A \otimes \rho^{BC}) \\ I(A:B) &= D(\rho^{AB}||\rho^A \otimes \rho^B) \end{aligned} \tag{1.2.57}$$

and the fact that quantum relative entropy is monotonic (1.2.43), so we can write that:

$$\begin{aligned} D(\rho^{ABC}||\rho^A \otimes \rho^{BC}) &\geq D(\rho^{AB}||\rho^A \otimes \rho^B) \\ I(A:BC) &\geq I(A:B) \\ I(A:B|C) &\geq 0 \end{aligned} \tag{1.2.58}$$

This inequality can also be expressed in the form:

$$S(AB) + S(BC) \geq S(B) + S(ABC) \tag{1.2.59}$$

which is mostly referred to as **strong subadditivity** of von Neumann entropy. Strong subadditivity is unique property of von Neumann entropy and can not be generalized to other definitions of quantum entropy. All of the existing proofs are very sophisticated and are out of the scope of this thesis. Interested reader can refer to [7] for the original proof or [6] for the proof of monotonicity of relative entropy.

1.2.7 Tripartite Information: Monogamy of Entanglement

Lastly, we define **tripartite information** [8], which is a nontrivial generalization of the mutual information:

$$\begin{aligned} I^3(A:B:C) &\equiv S(A)+S(B)+S(C) \\ &- S(AB)-S(BC)-S(AC)+S(ABC) \\ &= I(A:B)+I(A:C)-I(A:BC) \end{aligned} \tag{1.2.60}$$

The

first line makes it clear that I^3 is symmetric under permutations of its arguments. The second line hints that I^3 is some kind of measure for extensivity of mutual information. In a general quantum system I^3 can be either positive, negative, or zero, which will depend on the total density matrix ρ^{ABC}. An important case to consider is when $I^3 \leq 0$ or:

$$I(A:B)+I(A:C) \leq I(A:BC) \tag{1.2.61}$$

This equation indicates that correlations can not be shared between subsystems. Suppose A and B are maximally correlated then A can not have any correlation with subsystem C. It is well known that classical correlation can be shared between systems, thus this is a particular property of quantum correlations. This property of quantum correlations (given by 1.2.61) is also known as **monogamy of entanglement.**

Chapter 2

Black Holes and Information Paradox

The goal of this chapter is to formulate the black hole information paradox and discuss several proposals that claim to resolve it. We will start our discuss with the classical theory of black holes and their thermodynamic properties. Then, we will use semiclassical approach to investigate quantum properties of black holes and derive Hawking radiation. This will suggest that quantum black holes evaporate from pure states into mixed states, contradicting unitarity of the quantum mechanics. This contradiction is named information paradox. This will lead us to the phenomenological description of unitary black holes called black hole complementarity. However, we will see that a closer look at the postulates of black hole complementarity reveals another inconsistency, known as AMPS argument. AMPS argument claims that unitarity and the equivalence principle are not compatible and there has to be a firewall at the horizon. This argument will motivate the discussion of the next chapter, which is dedicated to the study of the link between entanglement and geometry. Existence of this powerful link can defeat the AMPS argument and bring old glory back to complementarity.

2.1 Classical Black Holes

Black holes are solution of the Eistein's equations, which correspond to the part of the space-time with very high concentration of matter density. The matter density is so high that nothing can escape it's gravitational force when crossing the horizon. In this section, we will discuss the simplest black hole solutions and their thermodynamic properties.

2.1.1 Schwarzschild metric

The solution of Einstein's equation corresponding to the exterior gravitational field of static, spherically symmetric object is given by Schwarzschild metric [9]:

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2 \qquad (2.1.1)$$

where t and r are time and radial coordinates respectively and Ω is the spherical angle. We notice that in $t - r$ coordinates the metric blows up for $r = 0$ and $r = 2GM$, the first corresponds to the singularity and second to the horizon. Metric blows up at Schwarzschild radius($r_s = 2GM$) due to the choice of coordinates, as t and r correspond to coordinates of outside observer who will never see anything crossing the horizon. To eliminate this problems let's first define tortoise coordinate:

$$r^* = r + r_s \ln \frac{r - r_s}{r_s} \qquad (2.1.2)$$

The next step is to define coordinates which are more naturally adapted to null geodesics. Let's then define:

$$\begin{aligned} u &= t - r^* \\ v &= r + r^* \end{aligned} \qquad (2.1.3)$$

In this case, the outgoing radial null geodesics are given by u = constant and infalling ones by v = constant. Thus, in new coordinates the Schwarzchild metric will be:

$$ds^2 = \frac{1}{2}(1 - \frac{2GM}{r})(dudv + dvdu) + r^2 d\Omega^2 \tag{2.1.4}$$

where r can be obtained from u and v coordinates. This choice of coordinates has few nice properties. First of all, none of the metric coefficients become infinite at $r_s = 2MG$. Second, light travels in $\pm 45^o$ angles, as u and v line correspond to null geodesics.

2.1.2 Extended Solution and Wormhole

Another possible choice of coordinate is called Kruskal-Szekeres coordinates, where U and V are defined as:

$$\begin{aligned} U &= -\exp(-\frac{u}{4GM}) \\ V &= \exp(\frac{v}{4GM}) \end{aligned} \tag{2.1.5}$$

In this coordinates, the Schwarzschild metric will be of the form:

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM} dUdV - r^2 d\Omega^2 \tag{2.1.6}$$

These coordinates possess all the nice properties of uv-coordinates and, in addition, the constant r curves for the outside observer are represented by hyperbolas in UV-plane and $UV = 0$ represents the horizons. Moreover, in UV-plane we can extend the Schwarzchild solution to include the negative V values. This is called maximally extended version of the Schwarzchild metric is drawn in Figure 2-1. Extended solution contains past singularity and another copy of the flat space. In the extended metric as well, light travels in $\pm 45^o$ angles and as illustrated in Figure 2-1 we be made of following regions:

Regions I is the original spacetime which is observable by physical instruments. It is our world.

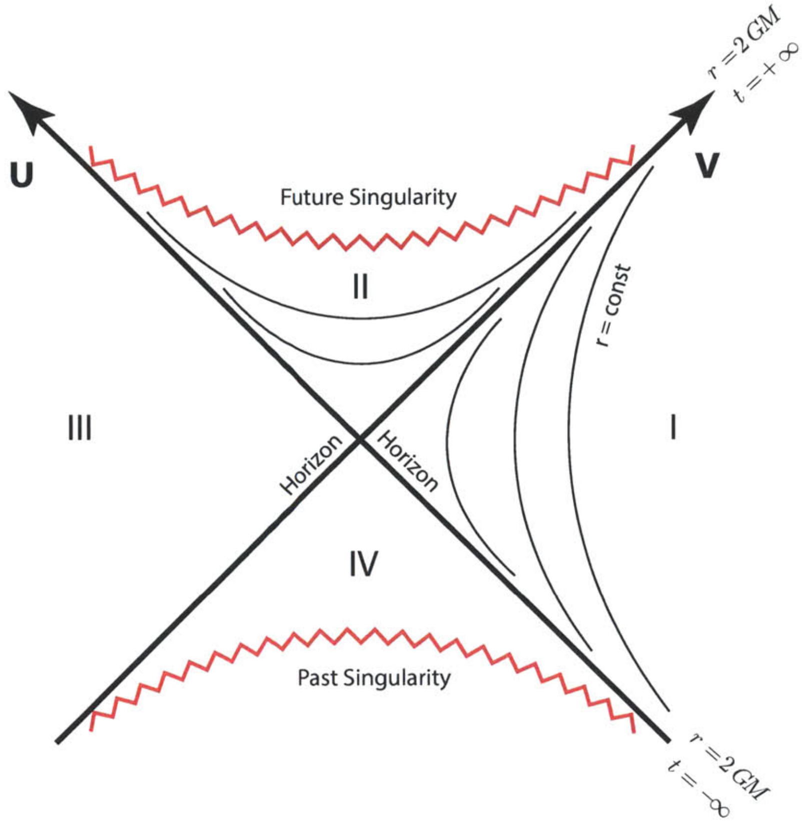

Figure 2-1: The maximal extended Schwarzschild spacetime in Kruskal-Szekeres coordinates.

Regions II describes the interior of the black hole. Infalling matter enters region II and will fall into the singularity at r = 0. Any light signal from region II will remain there and also fall in the singularity.

Regions III has properties identical with those of Region I.

Regions IV is the time reversal of Region II. An observer present in IV must have been originated in the past singularity and must leave Region IV to enter region I or III. Therefore IV is called a **white hole**.

An interesting property of extended solution, is that it contains so called Einstein-Rosen bridge (**wormhole**) between Regions I and III. This can be easily illustrated by embedding the solution into 3D flat space. Then, for t=0 the embedded solution will be made of two asymptotically flat surfaces that represent regions I and III ($r > r_s$), which are connected by a wormhole. Thus, all the regions with $r < r_s$ will be left out.

Figure 2-2 illustrates this embedded solution that can be pictured as 2-dimensional bridge between two flat geometries. Similar wormhole solutions exist for all the constant values of t coordinates and have corresponding geometrical representation as an embedding into higher dimensional space.

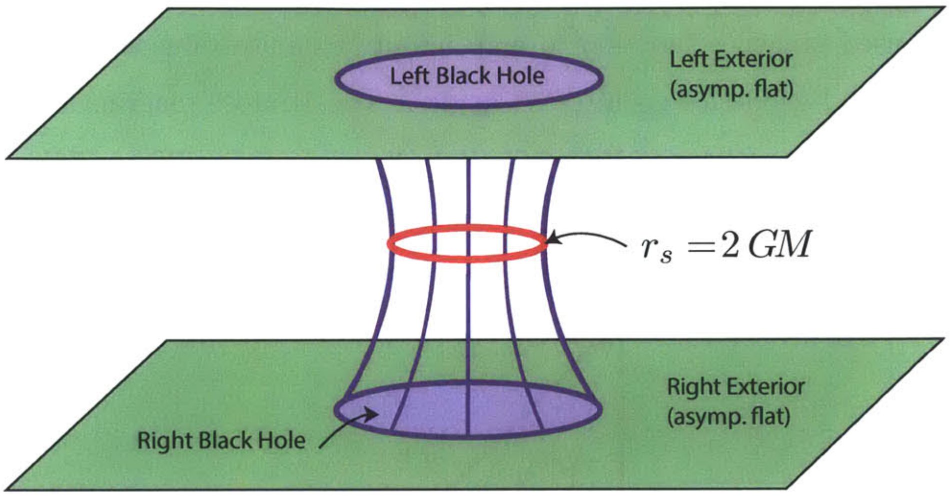

Figure 2-2: The spherical geometry of the hypersurface at t = 0 shown as it is embedded in flat space. The figure contains all space points $r \geq r_s$ but all points $r < r_s$ are lacking.

2.1.3 Penrose Diagrams

Most of the problems involving black holes use so called Penrose diagrams, which are a very convenient schematic way to represent spacetimes. To obtain a Penrose diagram of a given metric one has to perform two subsequent transformations. Firstly, do coordinate transformation ensuring that radial null geodesics lie at $\pm 45^o$. Secondly, perform a conformal transformation which respects angles but changes distances. Second transformation is done to bring infinity at finite distance so that a compact representation of the spacetime can be obtained. More precisely, all the points at infinity in the original metric should be at a finite value of the parameter in the new metric.

Minkowski spacetime

For the case of the flat space, we can define the coordinates ω, η as follows:

$$\begin{aligned} t-r &= \tan(\omega), \quad -\frac{\pi}{2} < \omega < \frac{\pi}{2} \\ t+r &= \tan(\eta), \quad -\frac{\pi}{2} < \eta < \frac{\pi}{2} \end{aligned} \tag{2.1.7}$$

where we used tangence function to map infinity to finite value and also it can be shown that null geodesics go in $\pm 45^o$ angles. The Penrose diagram of Minkowski space thus will have following form

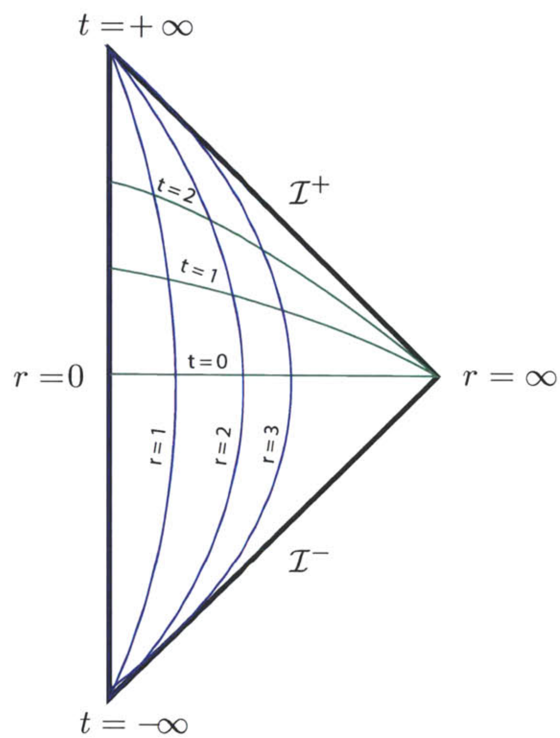

Figure 2-3: The Penrose diagram of the flat Minkowski spacetime. $\mathcal{I}^{\pm}$ denote the future/past null infinity, where both t and r coordinates diverge to infinity.

Schwarzchild spacetime

The case of the Schwarzchild metric is also similar, in this case $\{u, v\}$ coordinates already have the property that radial null geodesics lie at $\pm 45^o$ degree, so only a conformation transformation is applied taking the infinity at finite affine parameter.

This can be obtained by the use of tangent function:

$$
\begin{aligned}
u &= \tan(\omega), \quad -\frac{\pi}{2} < \omega < \frac{\pi}{2} \\
v &= \tan(\eta), \quad -\frac{\pi}{2} < \eta < \frac{\pi}{2}
\end{aligned}
\tag{2.1.8}
$$

Thus, in this new coordinates ω, η the maximally extended Schwarzchild solution can be represented as which is simply the confined version of the U, V representation

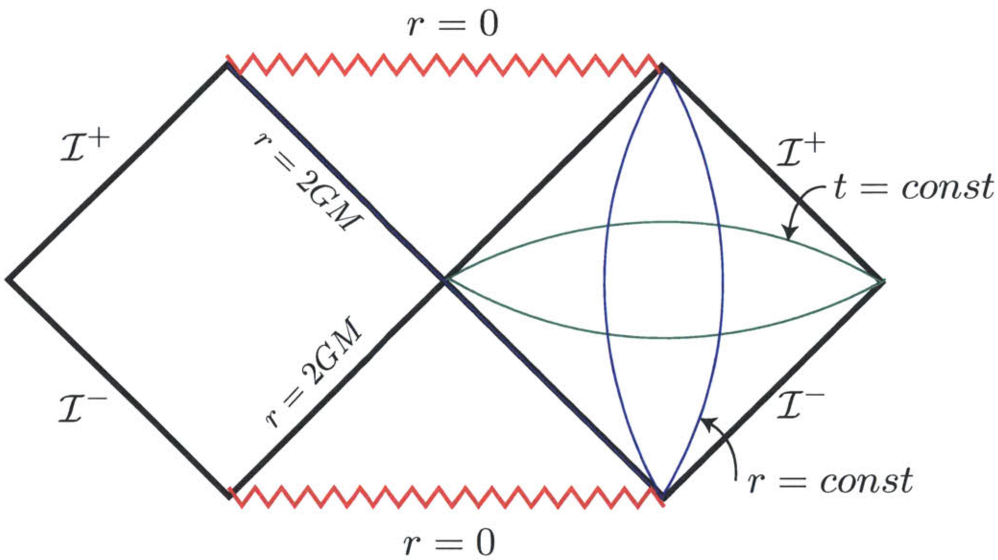

Figure 2-4: The Penrose diagram of the maximally extended Schwarzchild spacetime.

discussed in previous section. Thus, this extended solution will also Einstein-Rosen bridges (wormhole) connecting two parts of the spacetime. Figure 2-5 displays the Penrose diagram representation of this wormhole solutions for various spacelike slices, where the spacelike slices far away from horizon corresponds to the constant time slices.

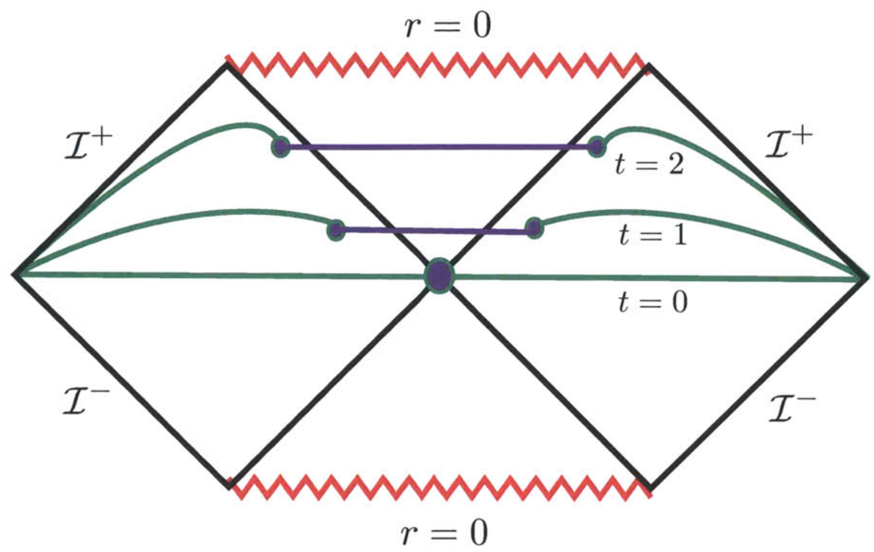

Figure 2-5: Penrose diagram representation of the wormhole solutions for various spacelike slices. Green lines correspond to the spacelike splice outside the horizons and purple line correspond to the wormhole inside the horizon.

2.1.4 Gravitational Collapse and "No hair" theorem

In a more realistic picture, black holes are created as a result of some gravitational collapse. To model this, we can consider the case of a spherical symmetric contracting shell of massless particles. This shell can be represented by an incoming light-like line in the Penrose diagram of Minkowski space (Figure 2-3). We also can draw the infalling radiation shell when the black hole was formed, in the Penrose diagram of the Schwarzchild metric (Figure 2-4). Lastly, we can appropriately much this two stages of the evolution of the shell and construct the total Penrose diagram, where initially we have flat space followed by the formation of the black hole. The gluing process is displayed in Figure 2-6: We can observe that the horizon will exist even before black hole is actually formed and black hole is formed when collapsing matter crosses this horizon. Figure 2-6 displays horizon with dashed line. When collapsing matter crosses the horizon and forms the black hole, we loose all the information about what made the black hole. In his book [10], physicist John Archibald Wheeler

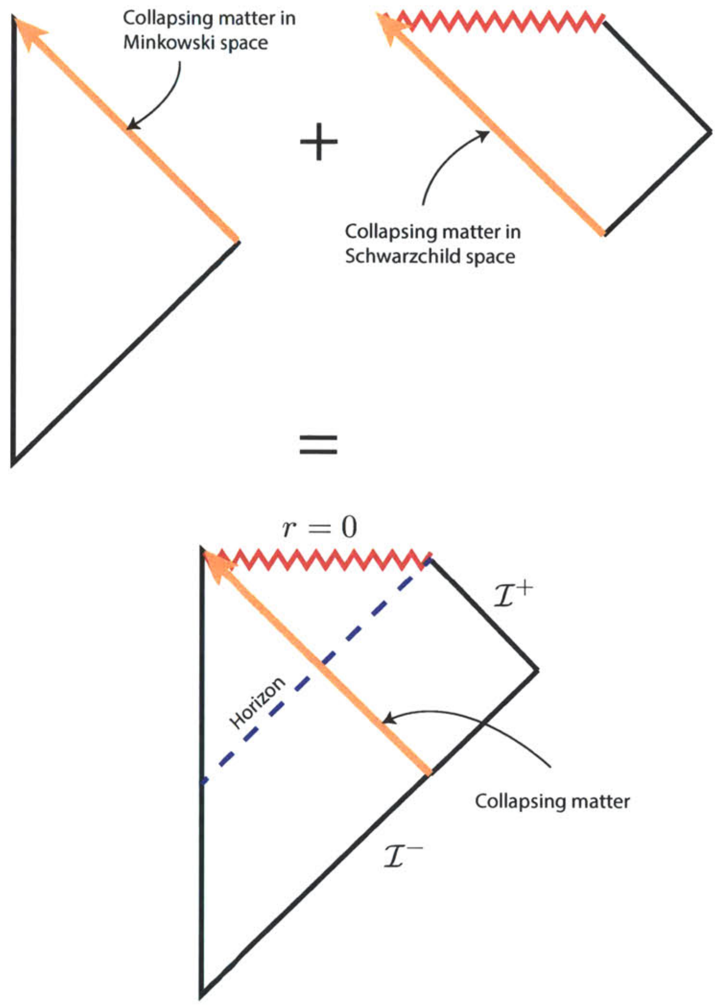

Figure 2-6: The construction of the spacetime of a gravitational collapse to a black hole.

mentioned that "black holes have no hair", referring to the idea that classical black holes have no information about the matter that formed it. This is due to the fact that horizon of the black hole is a perfect sphere and any deviation from this will be quickly absorbed. Thus, any information that crossed the horizon is permanently inaccessible to external observer and black hole is characterized only by it's mass, charge and spin. No hair theorem has been proved to be true for several examples [11], but the most general case remains a conjecture.

2.1.5 Thermodynamics

Early on in the study of classical black holes quite striking analogy was discovered between black hole mechanics and laws of thermodynamics. In particular, physicists noticed that if one tries to solve Einstein's equations in an asymptotically flat space-time which is stationary, one finds that [12]:

Zeroth Law *The surface gravity k is constant on the future event horizon of a stationary black hole.*

This is analogous to the zeroth law of thermodynamics which states that the temperature is constant throughout a system in thermal equilibrium. This suggests some kind of similarity between the surface gravity and temperature. Next, for small perturbations of stationary black holes, one can calculate the change of energy and find following relationship [12]:

$$\frac{k}{8\pi G}\Delta A = \Delta M - \Omega_H \Delta J \tag{2.1.9}$$

This is called the **First law** of black hole mechanics, as it is very similar to the first law of thermodynamics:

$$T\Delta S = \Delta E + P\Delta V \tag{2.1.10}$$

Identification of the two laws leads to the conclusion that the black hole mass plays the role of energy in the ordinary first law and $-\Omega_H \Delta J$ represents the work term. From this identification, one can also recognize the connection of black hole horizon area with the entropy, known as Bekenstein-Hawking entropy [13]:

$$S_{BH} = \frac{1}{4G} A \propto M^2 \tag{2.1.11}$$

implying that $k/2\pi$ should take the role of the temperature, which is enforced by the zeroth law. The fact that black holes have entropy makes it clear that second law of thermodynamics is not violated when something is thrown into black hole, as it will increase the mass of the black hole and so it's entropy. Analogous to second law of thermodynamics is **area law** of black holes, which states that area of the black hole never decreases, assuming weak energy condition. This topic is out of the scope of this thesis and will not be address later in the discussion. One can also show that there are no black holes with vanishing surface gravity ($k = 0$). This is known as the **Third law** and is analogous to the third law of thermodynamics which states that no physical system has zero temperature. Some physicists may argue that extremal black holes have vanishing surface gravity, but it was shown that the backreaction conspires to prevent the black hole from actually reaching the extremal limit.

2.2 Quantum Theory of Black Holes

In this section, we will use semiclassical approach to study quantum mechanical properties of black holes. Semiclassical approach to quantum field theory in curved spacetime is a theory where gravity is treated in general relativistic way, but matter is treated according to the laws of quantum field theory. It is known that this combination does not serve as an accurate description of nature, but is a good approximation when we are far away from Planckian scales.

2.2.1 Hawking Radiation

In his article [14] Hawking argued that this approach can be used to study behavior of the quantum field near the black hole horizon, which led to the derivation of the Hawking radiation. His argument was based on the fact that the radius of curvature of space-time outside the event horizon is very large compared to the Planck length $(G\hbar/c^3)1/210^{33}$ cm, the length scale on which quantum fluctuations of the metric are expected to be of order unity. Using semiclassical approach, Hawking showed that doing quantum field theory in curved space will lead to a particle creation near horizon, when one of the particles is emitted away from the black hole and the other is drawn into it. This particle creation is very similar to electron-positron pair creation in magnetic field, when in the presence of the strong field particles do not get created and annihilated immediately, but rather are drawn to opposite directions by the field. This calculation can be done carefully [15] and one finds that state of the field near horizon dependents on the choice of the time coordinate, as in the curved space there are numerous ways one can choose time coordinate (Figure 2-7). In particular, for the

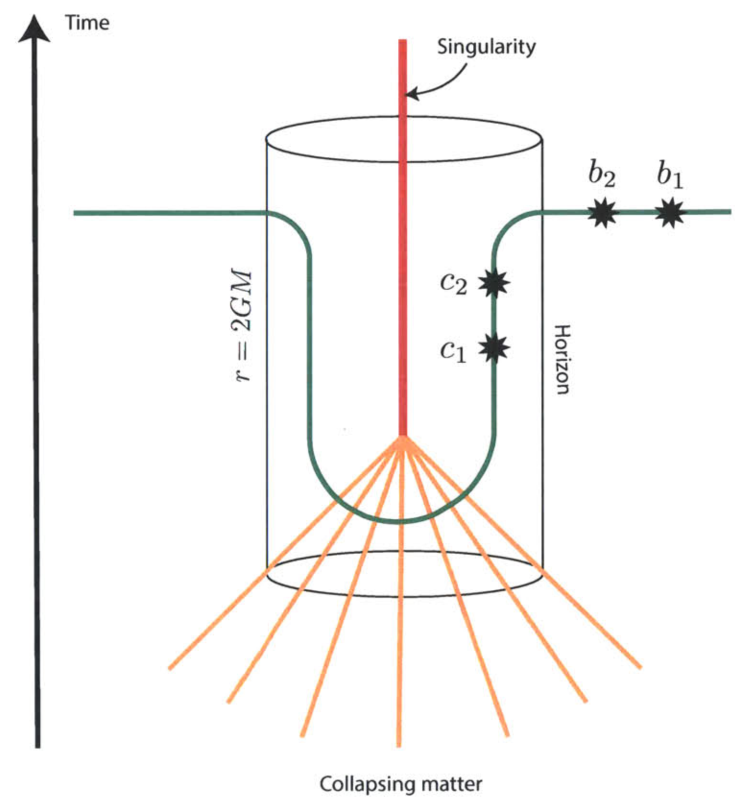

Figure 2-7: Space-like curve corresponds to constant t coordinate outside the black hole and constant r inside black hole, which makes a turn at the horizon. On this space-like curve two pairs of particles and one can observe that pair (b_2, c_2) was created after (b_1, c_1) pair.

choice of the time coordinate that asymptotes to the time coordinate of the exterior Minkowski space, the state of the field on a space-like slice with fixed time coordinate t_1 outside the horizon will be given by [15]:

$$|\psi\rangle_1 = Ce^{\gamma \hat{b}_1^\dagger \hat{c}_1^\dagger}|0\rangle_{b_1}|0\rangle_{c_1} \tag{2.2.12}$$

where C is a constant, $|0\rangle_{b_1}$ and $|0\rangle_{c_2}$ are the vacuum states of the quantum fields inside and outside horizon for the space-like slice Figure 2-7, $\hat{b}_1^\dagger$, $\hat{c}_1^\dagger$ are the particle creation operators acting on those vacuum states. This particles are created for all of the space-like slices and each newly created state is independent of the previously created particle pairs. Figure 2-7 can help you to visualize particle creation process of two consecutive Hawking pairs. Taylor expanding the state $|\psi_1\rangle$:

$$\begin{aligned} |\psi\rangle_1 &= C\left(|0\rangle_{b_1} \otimes |0\rangle_{c_1} + \gamma \hat{b}_1^\dagger |0\rangle_{b_1} \otimes \hat{c}_1^\dagger |0\rangle_{b_1} + \frac{\gamma^2}{2} \hat{b}_1^\dagger \hat{b}_1^\dagger |0\rangle_{b_1} \otimes \hat{c}_1^\dagger \hat{c}_1^\dagger |0\rangle_{b_1} + ...\right) \\ &= C\left(|0\rangle_{b_1} \otimes |0\rangle_{c_1} + \gamma |1\rangle_{b_1} \otimes |1\rangle_{c_1} + \frac{\gamma^2}{2} |2\rangle_{b_1} \otimes |2\rangle_{c_1} + ...\right) \end{aligned} \tag{2.2.13}$$

where $|n\rangle_{b_1}$ means that we have n quanta of type b_1 in the state. The most important feature of this Hawking state is that the b_1 and c_1 excitations are *entangled.* The number γ is order unity, so we only need to look into first few terms of the Taylor expansion. Thus, to understand the significance of the entangled nature of the state we can replace the state in Equation (2.2.13) by the simpler state:

$$|\psi\rangle_1 = \frac{1}{2}\left(|0\rangle_{b_1} \otimes |0\rangle_{c_1} + |1\rangle_{b_1} \otimes |1\rangle_{c_1}\right) \tag{2.2.14}$$

This is the maximally entangled state shared by Alice and Bob that we talked about in Section 1.1.4 of Chapter 1. In Section 1.1.7 we showed that the reduced density matrix for the individual qubit will be fully mixed. Thus, the observer sitting outside the black hole and measuring quanta $b_1, b_2, b_3, ...$ can never identify the pure state of the $(b_1, c_1), (b_2, c_2), ...$, but will instead see quanta in a mixed state. This is due to the fact that each of (b_k, c_k) lives at a location different from the other pairs, so the overall state will be a direct product of states for each of these pairs [15]. Due to this evaporation process the mass of the black hole will decrease, but semiclassical approach will hold true until the size of the black hole reaches Plackian limit. It turns out that the temperature corresponding to this Hawking radiation is equal to the thermodynamics temperature from Section 2.1.5, which is equal to $k/2\pi$. Thus, we can give an estimate to the total entanglement entropy of the emitted quanta. To do it we first use the fact that the entanglement entropy of the pair (b_k, c_k) is about 1 bit and the energy of each quanta is calculated to be $(GM)^{-1}$ [14]. Thus, dividing total mass M over the energy of single quanta and multiplying by 1 bit we get:

$$S_{ent} \propto M(GM) \propto M^2 \tag{2.2.15}$$

which is of the order of the thermodynamic entropy of the black hole discussed in Section 2.1.5. More careful calculations have been done to identify coefficients for the entanglement entropy. The exact correspondence between this two quantities was established in [16, 17], when the authors took into account the backreaction effect of the radiation and modeled the radiation process as a tunneling process. To summarize, semiclassical approach to black holes suggests that black holes evaporate and after evaporation, the emitted radiation is in an entangled state with the remnant of the black hole. Also, if we wait long for the black hole to evaporate up to Planck size, the entanglement entropy becomes equal to Bekenstein-Hawking entropy of the initial black hole.

2.2.2 Information Paradox

Everything we discussed until now seems reasonable and justified, but it turns out that if one looks into a bigger picture of the black hole formation and evaporation there is an inconsistency in those theories. Imagine one decides to collapse a sphere of radiation in a pure state $|\psi\rangle$ into a black hole and waits for it to fully evaporate. Then, according to the discussion in Section 2.2.1 the evaporated Hawking radiation will be in a mixed state with entropy S_{ent}. Thus, information is lost during this process and evolution is not unitary, which is in conflict with the principles of quantum mechanics. This contradiction is known as:

Information Paradox *Modeling the black hole evolution with the principles of general relativity and quantum theory leads to a contradiction, as these principles prohibit the evolution of a pure state to a mixed state.*

Information is conserved in quantum mechanics, it is never created and never destroyed as the evolution is unitary. It is generally accepted that quantum gravity should also be unitary and, thus, it is expected that effective quantum theory of black holes should also conserve information. Moreover, recent discovery of the AdS/CFT correspondence [18, 19] provides an evidence of such unitary dynamics in string theory. In the AdS/CFT picture, the black hole in AdS space corresponds to a state of conformal field in the boundary and one can show that because dynamics in CFT is unitary so must be the dynamics of black hole in AdS space. The fact that semiclassical approach leads to the non-unitary dynamics can mean two things either there is something wrong with the assumptions of the semiclassical calculation and corrections need to be made, or quantum gravity is not unitary. In my opinion, second scenario is highly unlikely, so further research needs to be done on the models of black hole formation and evaporation to resolve this paradox. It is indeed possible, that we might need quantum gravity to provide such mechanism, but before that we can use this paradox to learn more about existing conflicts between gravity and quantum theory.

2.2.3 AdS/CFT correspondence

The Ads/CFT correspondence, which argues that (quantum) gravity in the $(d+2)$-dimensional anti de-Sitter space AdS_{d+2} is equivalent to a $(d+1)$-dimensional conformal field theory CFT_{d+1}[18]. The Poincare metric for AdS_{d+2} with radius R is:

$$ds^2 = R^2 \frac{dz^2 - dx_0^2 + \sum_{i=1}^{d-1} dx_i^2}{z^2} \tag{2.2.16}$$

where the dual CFT_{d+1} lives on the boundary of AdS_{d+2} which is a flat space at $z \to 0$ spanned by the coordinates (x_0, x_i). Since the metric diverges in the limit $z \to 0$, we put a cut off by imposing $z \geq a$. Then the boundary is located at $z = a$ and is identified with the ultraviolet cut off in the dual CFT. Thus, a fundamental principle of AdS/CFT [20, 18], known as the bulk to boundary relation is simply expressed by the equivalence of the partition functions in both:

$$Z_{CFT} = Z_{AdS-Gravity} \tag{2.2.17}$$

So, perturbations in the AdS background are dual to the shift of background in the CFT side and we can compute the correlation functions in the CFT by taking the derivatives with respect to the perturbations in AdS. This also implies that because dynamics in CFT is unitary so must be dynamics in AdS space. So far we applied the AdS/CFT to the pure AdS spacetime (2.2.16). However, the AdS/CFT can be applied to any asymptotically AdS spacetimes including the AdS black holes. One could argue that the information paradox is resolved by the discovery of AdS/CFT, but is naive statement. The known agreements between AdS gravity and the CFT involves comparison of scaling dimensions, n-point correlations, which merely confirms the fact that quantum gravity is unitary. However, AdS/CFT does not tell anything about black hole formation and evaporation process to resolve the information paradox. To solve the information paradox, one has to provide a mechanism to get the information out of the black hole.

2.2.4 Complementarity

In this section, we will not directly address the contradiction between semiclassical approach and unitarity, but instead, we will assume that black holes are governed by entirely unitary dynamics and use this assumption to gain more insight into the structure of quantum black holes. Here we will assume that state vectors on one space-like slice is evolved to future space-like slice by a linear operation (S). The full dynamics of black hole formation and evaporation is presented in Figure 2-8, where $|\Psi(\Sigma)\rangle$ denotes the state on some slice Σ. This state can be linearly evolved until the surface Σ_P is reached, which is the surface containing point P where horizon and singularity meet. Let's denote by Σ_{bh} and Σ_{out} the parts of the Σ_P that lie inside and outside the black hole horizon respectively. Assuming linearity of the entire process,

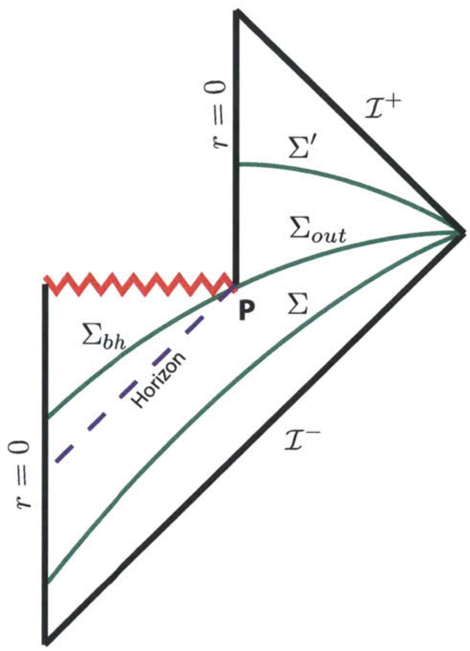

Figure 2-8: Dynamics of the spacelike slices during the black hole formation and evaporation. Σ represents the slice when black hole wasn't formed yet. $\Sigma_P = \Sigma_{bh} \cup \Sigma_{out}$ is the slice that contains point P. And Σ' is the slice after black hole have evaporated.

the post-evaporation state on slice Σ' will be pure and can be denoted by:

$$|\Psi(\Sigma')\rangle = \mathbf{S}_1 |\Psi(\Sigma)\rangle \tag{2.2.18}$$

On the other hand, if we look into the dynamics following the complete evaporation, the state on Σ_{out} is the one that was linearly evolved to $|\Psi(\Sigma')\rangle$, thus it must also be pure state ($|\Phi(\Sigma_{out})\rangle$ and:

$$|\Psi(\Sigma')\rangle = \mathbf{S}_2 |\Phi(\Sigma_{out})\rangle \tag{2.2.19}$$

Thus, the total state on Σ_P will be the tensor product of two pure states defined on Σ_{bh} and Σ_{out}:

$$|\Psi(\Sigma_P)\rangle = |\chi(\Sigma_{bh})\rangle \otimes |\Phi(\Sigma_{out})\rangle \tag{2.2.20}$$

Using this and the fact that state $|\Psi(\Sigma_P)\rangle$ is the linear evolution of state $|\Psi(\Sigma_P)\rangle$, we obtain the following relation:

$$|\chi(\Sigma_{bh})\rangle \otimes |\Phi(\Sigma_{out})\rangle = \mathbf{S}_3|\Psi(\Sigma)\rangle \tag{2.2.21}$$

Equations (2.2.18), (2.2.19) and (2.2.21) taken together indicate that the state inside the black hole must be independent of the initial state $|\Psi(\Sigma)\rangle$. This result is consistent with the intuitive idea that the observer outside the black hole never sees the state crossing the horizon, but rather sees it freezing on the surface, scrambling and in the end, evaporating in a form of radiation. This makes quite a lot of sense, even if we do not really have a mechanism that explains the entire evaporation process. This explains the point of view of the outside observer, but according to equivalence principle infalling observer does not see anything special at the horizon thus (s)he should see the total state evolving appropriately into the black hole. So, what is really happening? A possible explanation is given by black hole complementarity, which is based on the idea that **no observer ever witnesses a violation of the laws of physics**, even if their observations are inconsistent with each other. This idea was rigorously formulated in [21] via a set of 4 postulates:

- **Postulate 1:** *The process of formation and evaporation of a black hole, as viewed by a distant observer, can be described entirely within the context of standard quantum theory. In particular, there exists a unitary S-matrix which describes the evolution from infalling matter to outgoing Hawking-like radiation.*

- **Postulate 2:** *Outside the stretched horizon of a massive black hole, physics can be described to good approximation by a set of semiclassical field equations.*

- **Postulate 3:** *To a distant observer, a black hole appears to be a quantum system with discrete energy levels. The dimension of the subspace of states describing a black hole of mass M is the exponential of the black hole entropy.*

- **Postulate 4:** *A freely falling observer experiences nothing out of the ordinary when crossing the horizon.*

We did not talk about the stretched horizon before, but one can think of it as an apparent horizon that is Planck length away from the real event horizon that is seen by the outside observer. It is an important cutoff for the observer who uses the semiclassical approach to describe the Hawking radiation. In a simpler language, postulate 1, 2 and 3 tell us that the outside observer can treat black hole as a large, complex, but conventionally quantum object which is evaporating out through Hawking radiation. The time it takes for a black hole, starting from its initial state, to reach the point where it starts to release its information is called Page time [22]. A black hole that has already past its Page time is called an old black hole. Postulate 4, on the other hand, is basically a formulation of the equivalence principle, that claims that infalling observer will not see anything special at the horizon.

2.2.5 AMPS argument and Firewalls

In previous section we introduced complementarity principle, which provides a framework for combining unitarity of quantum mechanics and equivalence principle of general relativity. In 2012, Almheiri, Marolf, Polchinski and Sully, came up with an argument widely known as AMPS argument [23], suggesting that these two fundamental principles are inconsistent and cannot be combined within the framework of black hole complementarity. They proposed to consider the black hole evaporation process after it reached the Page time, when information starts leaking out of the black hole. The situation is presented in Figure 2-9, the black hole is maximally entangled with the early Hawking radiation R that has already been emitted. Let's denote by O the next Hawking quanta that gets emitted and by I it's partner interior mode. Assuming that system R, I and O are independent we can write strong subadditivity of the von Neumann entropy discussed in the Section 1.2.6:

$$S(RO) + S(OI) \geq S(O) + S(ROI) \tag{2.2.22}$$

On the one hand, assuming the whole process is unitary we have that after Page time the entropy of the early radiation R has to decrease when O is emitted, because the

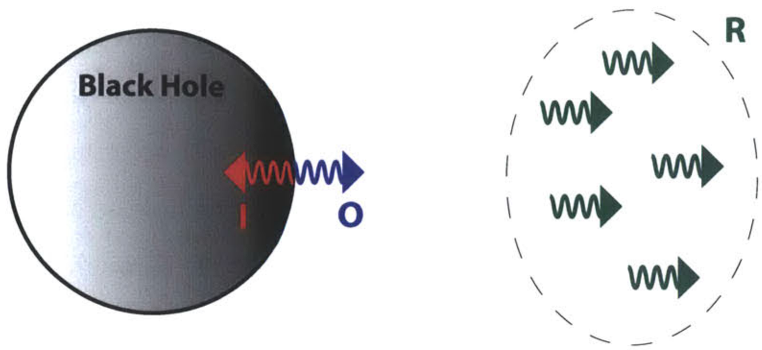

Figure 2-9: Black hole evaporation after Page time. R is the early Hawking radiation, O is an outside mode of just emitted radiation and I is the partner interior mode of O.

mixed state of radiation is becoming pure. Thus, after Page time entropy of the total radiation is decreasing [22]:

$$S(OR) < S(R) \tag{2.2.23}$$

On the other hand, according to equivalence principle the infalling observer experiences the vacuum state, which means that outgoing quantum O and interior mode I are in a maximally entangled pure state. Thus, we conclude that $S(IO) = 0$, $S(O) > 0$ and $S(ROI) = S(R)$, which combined with Equation (2.2.22) gives:

$$S(RO) > S(R) \tag{2.2.24}$$

Equation (2.2.23) and (2.2.24) are in direct contradiction and create an important challenge for black hole complementarity. In short AMPS argument says that for an infalling observer to experience the horizon as harmless place the outgoing and interior modes have to be maximally entangled. And for the evolution to be unitary, the entanglement entropy of the black hole has to decrease after the Page time, which is true only if the outgoing mode is entangled with the early radiation. The monogamy of entanglement discussed in Section 1.2.61 prevents this from happening. Furthermore, Figure 2-10 suggests that it is possible to design an experiment when infalling observer Bob can learn about maximal entanglement of O with R and O with I reaching the

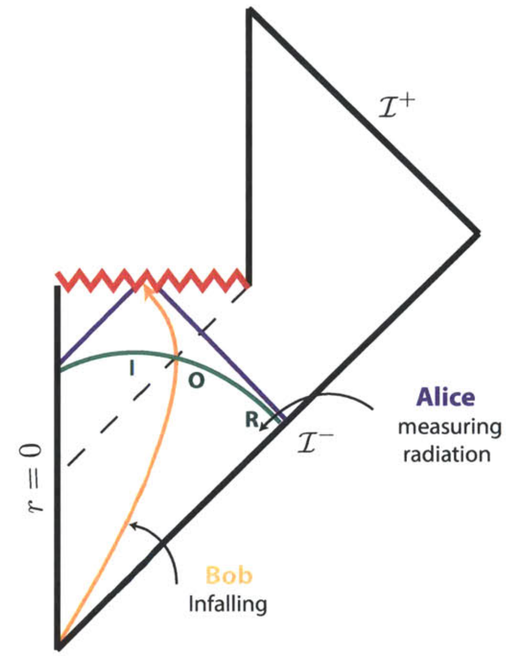

Figure 2-10: Black hole evaporation after Page time. R is the early Hawking radiation, O is an outside mode of just emitted radiation and I is the partner interior mode of O.

singularity. In this scenario, Bob measures entanglement between O and I himself, but learns about the entanglement of O with R from a signal that Alice sent from outside after doing measurement on R. This contradicts complementarity, as an infalling observer Bob will see violation of physics law (monogamy of entanglement). Authors of the argument proposed a possible resolution to it [23], suggesting that equivalence principle doesn't hold and horizon is not a harmless place for the infalling observer. They proposed that infalling Bob will hit a firewall at the event horizon that will burn him to death. This suggestion received significant amount of attention from physics community [24, 25, 26, 27], but it is out of the scope of this thesis. Instead of talking about feasibility of firewalls, in the next chapter we will focus on a conjecture by Maldacena and Susskind that provides a possible resolution to this argument by challenging the assumption that early Hawking radiation(R) and interior mode (I) are independent, which is necessary for the correctness of the strong subadditivity. In particular, they propose that early Hawking radiation and interior of the black hole are connected by a wormhole and Bob will see firewall at the horizon if and only if Alice performs measurements on the early Hawking radiation, thus sending the firewall down the wormhole. We will explore this connection between entanglement and wormholes in a greater detail in the next chapter.

Chapter 3

The ER=EPR conjecture

3.1 Introduction

It has been suggested that spacetime has an underlying, quantum information theoretic description [28]. The ER = EPR conjecture [29] is a specific realization of this proposal. This conjecture was initially proposed to address the AMPS argument, resolving (or at least clarifying) the conflict between unitarity, equivalence and locality. It is partly based on intuition that wormhole in AdS space corresponds to a specific entangled state in CFT. The conjecture generalizes this, saying that the quantum degrees of freedom corresponding to black holes connected by an Einstein-Rosen (ER) bridge [30] are entangled. Furthermore, the entanglement entropy is related to the cross sectional area of the bridge. The conjecture is summarized as ER = EPR, where EPR stands for Einstein, Podolsky, and Rosen [31] and represents entanglement.

In this chapter, I will motivate and present the ER = EPR conjecture, relating it to the discussion of previous chapters.. Then will focus on a paper I co-authored with my friend Bob Penna [32], were we present a specific test of the conjecture. We show that the geometrical notion of entropy associated to ER bridges obeys the same inequalities as ordinary entanglement entropy: subadditivity, the triangle inequality, strong subadditivity, and a set of inequalities discovered by Cadney, Linden and Winter (CLW) [33, 34]. This is evidence for the ER = EPR conjecture.

We also describe a restriction the conjecture imposes on the entangled degrees of freedom underlying classical ER bridges. The entanglement must have nonpositive tripartite information [35]. This is not a general property of quantum systems. For instance, the four qubit pure state $|GHZ_4\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}$ has positive tripartite information. Macroscopic black holes can be built out of many copies of $|GHZ_4\rangle$ which do not have classical ER bridges. This serves to emphasize that the ER bridge depends on the pattern of entanglement and not just the total amount of entanglement.

3.2 AdS Back Holes

The relationship between wormhole and entanglement is most rigorously understood in the ADS/CFT framework discussed in Section 2.2.3 [18]. Consider the eternal AdS-Schwarzschild black hole whose Penrose diagram is shown in Figure 3-1. This diagram displays the two exterior AdS regions and two interior regions, one being the interior of the black hole and the other is white hole. L and R denote the boundaries of left and right exterior AdS regions respectively. In Section 2.1.2, we have seen that the extended solution can be interpreted as two black holes in disconnected spaces with a common time that are connected by Einstein-Rosen bridge. The green line denotes the constant $t = 0$ spacelike slice and can be interpreted as two AdS exteriors connected by a wormhole: According to AdS/CFT correspondence shows this black hole solution in the AdS space corresponds to special state $|\Psi\rangle$ defined on left and right CFT's [36]:

$$|\Psi\rangle = \sum_n e^{-\beta E_n/2}|n\rangle_L|n\rangle_R \qquad (3.2.1)$$

where E_n denote the discrete energy levels of CFT's on the boundary, with corresponding eigenstates $|n\rangle_L$ and $|n\rangle_R$. This state illustrates that the two CFT's corresponding to each of the black holes are in a highly entangled state and the entanglement entropy was calculated to be equal to the Bekenstein-Hawking (2.1.11) entropy of either black hole. This example of thermal state makes a powerful case for the correspondence between entanglement and Eistein-Rosen bridges. Moreover,

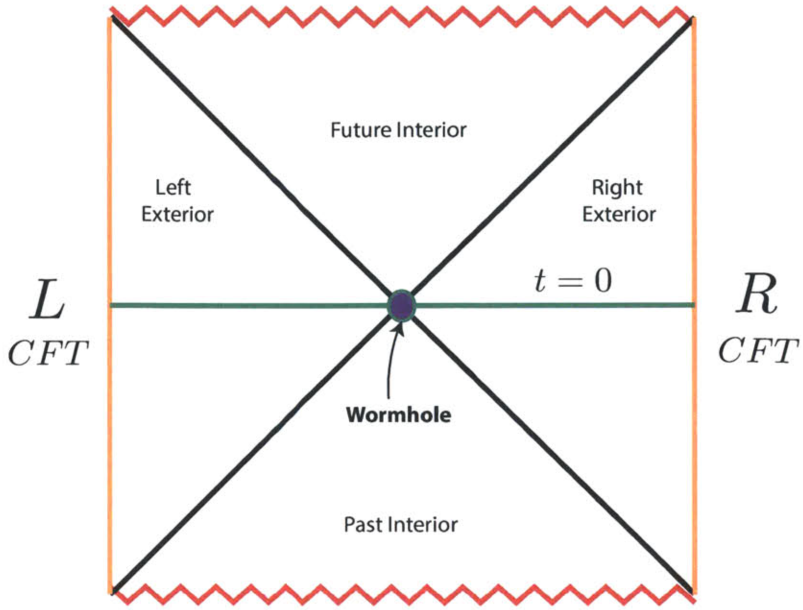

Figure 3-1: Penrose diagram of the AdS black hole. L and R denote the left and right CFT's living in the boundary of AdS. The green line represents the $t = 0$ slice, where left and right exteriors are connected by a wormhole.

one can think of the entanglement between left and right CFT's as a representation of the entanglement between the black holes themselves and generalize this idea to the case of astrophysical black holes.

3.3 Holographic Entanglement Entropy

As we have seen in previous section, the entanglement entropy of the thermal state (3.2.1) is proportional to the area of the black hole horizon. Motivated by this correspondence Ryu and Takayanagi [37] proposed a more general formula for calculating entanglement entropy for a region of the CFT. Suppose, we divide the AdS_{d+2} boundary time slice N into regions A and B Figure 3-2. In the Poincare coordinate (2.2.16), the CFT_{d+1} is supposed to live on the boundary $z = a \to 0$ of AdS_{d+2}. To have its dual gravity picture, we extend this division $N = A \cup B$ to the time slice M of the bulk spacetime. If we denote by ∂A boundary of region A, then we extend ∂A to a surface γ_A in the entire M such that $\partial\gamma_A = \partial A$. There are infinitely many choices of γ_A, but according to [37] we have to choose the unique surface that has minimal area and call it γ_A. They further propose that entanglement entropy S_A of region A in CFT_{d+1} is given by formula:

$$S_A = \frac{Area(\gamma_A)}{4G_N^{d+2}} \tag{3.3.2}$$

where γ_A is the minimal area surface in AdS_{d+2} whose boundary is ∂A and G_N^{d+2} is the $(d+2)$-dimensional Newton constant of the AdS gravity. See Figure 3-2 for visualization. We can regard the formula (3.3.2) as a generalization of (2.1.11), as

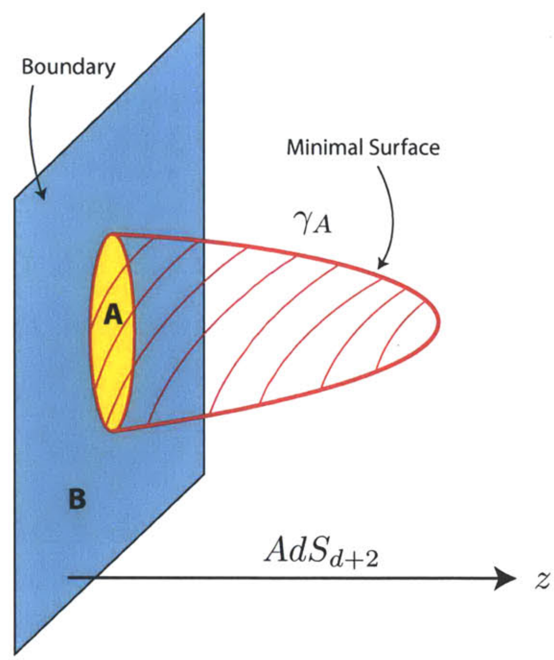

Figure 3-2: The holographic calculation of entanglement entropy via AdS/CFT.

in the presence of event horizon the minimal surface will wrap around the horizon. Figure 3-3 illustrates various cases of the minimal area cut in the case of the 2-dimensional AdS black holes and proves that Bekenstein-Hawking formula is obtained if $A = N$ and γ_A wraps the horizon.

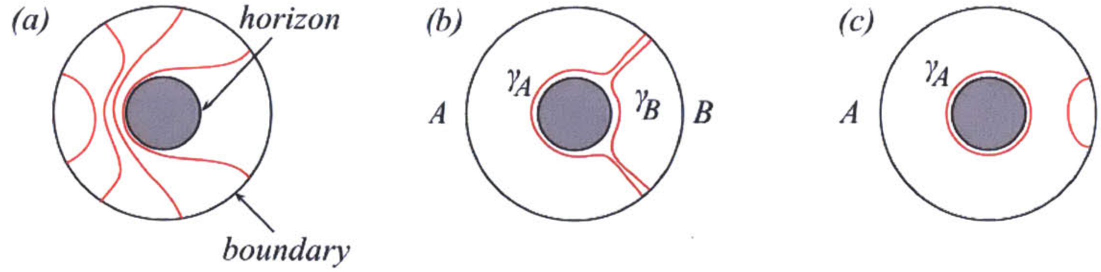

Figure 3-3: (a) Minimal surfaces γ_A in the black hole for various A. (b) γ_A and γ_B wrap the different parts of the horizon. (c) When A gets larger, A is separated into two parts: one is wrapped on the horizon and the other localized near the boundary.

3.4 Schwarzchild Black Holes and AMPS

Maldacena and Susskind [29] use the results for AdS black holes to provide an interpretation to extended Penrose diagram of the Schwarzchild black hole (2-4). In this case, we have two spatial sheets that are asymptotically flat and each has an identical black hole. The green line in FigSC-wormhole denotes the state at $t = 0$ and the flat exteriors regions are connected by the wormhole, because two horizons touch at t=0. Generalizing the result from AdS black holes in Section 3.2, we conclude that those black holes must be in an entangled state. In the embedded diagram Figure 2-2, although the geometry is connected through the bridge, the two exterior regions are not in causal contact and no information flows across the bridge. This can be seen from the Penrose diagram (2-4) and is consistent with the fact that entanglement does not support a non-local signal propagation. This interpretation is well studied theoretical topic, but has almost no applications to astrophysical black holes, as exterior regions are causally disconnected. It turns out, that there exists an alternative interpretation of extended Schwarzschild metric that can have meaningful astrophysical consequences. In this interpretation, instead of black holes on two disconnected sheets, we can consider two very distant black holes living in the same space. Assuming the duality between entanglement and wormhole, we conjecture that if the black holes are somehow created in the entangled state, then they will be connected by a wormhole. Otherwise, they would not have a wormhole connecting them. See Figure 3-6 for this interpretation. This interpretation is the key idea be-

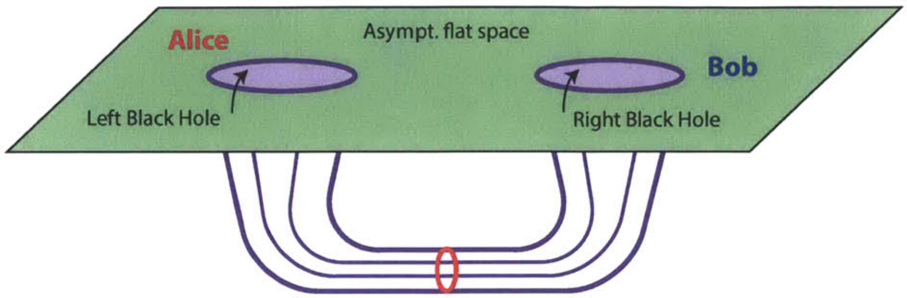

Figure 3-4: Extended solution of the Schwarzchild black hole can be interpreted as two black holes in the same space located far away from each other, but connected by a wormhole. Minimal radius of the wormhole depends on the choice of the spacelike slice.

hind ER = EPR conjecture, as it uses the abstract relation between entanglement and geometry from AdS/CFT to construct physically addressable question in Minkowski space. Moreover, this conjecture directly addresses the AMPS argument presented in Section 2.2.5. Suppose Alice is located near Left black hole and Bob is near the

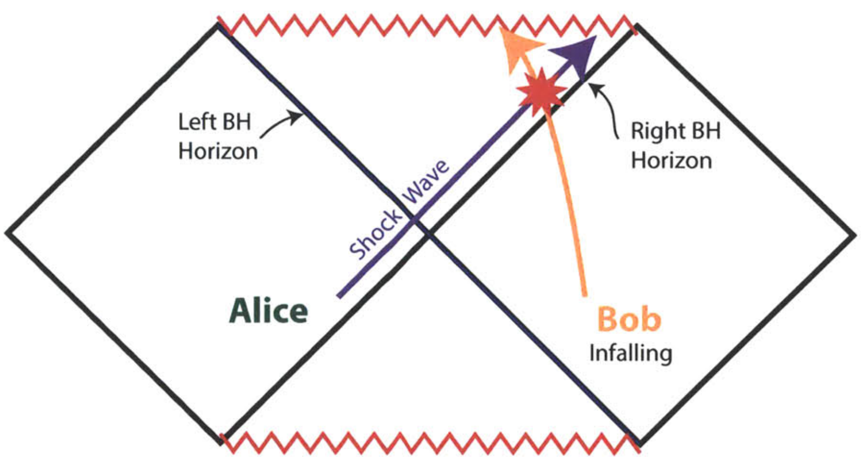

Figure 3-5: Penrose diagram for two far away black holes in the Minkowski space that are connected by a wormhole. Bob is falling into right black hole. Alice is sending a shock wave into left black hole. Bob is burned by a shock wave at the horizon. Thus the degrees of freedom inside Bob's black hole are not independent of the degrees of freedom outside Alice's black hole.

Right one. Left sand Right is a conventional choice for labeling the black holes that are very far away and are connected by a wormhole. Then, if Bob and Alice stay outside their respective black hole horizons, communication between them can only take place through the long trip in the exterior space. On the other hand, Bob and Alice can jump into their respective black holes and meet very soon in the interior, because black holes are connected by a wormhole (Figure 3-5). Thus, we can also conclude that for this case if Bob jumps into the black hole what he will see inside the horizon will greatly depend on what will Alice do to her black hole. In particular, Alice can create a firewall on Bobs side if she throws in shock waves into her black hole early enough. This property of Einstein-Rosen bridges can provide a clarification

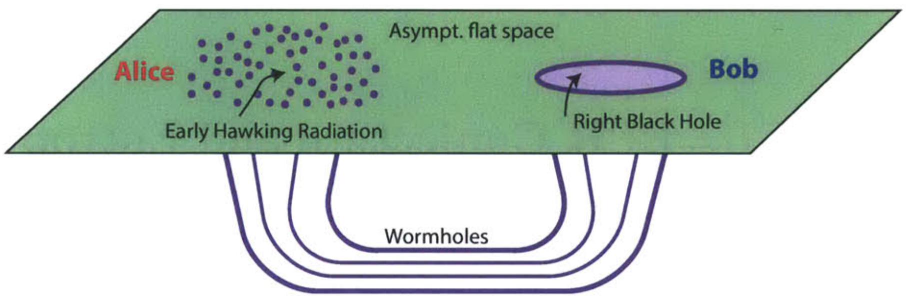

Figure 3-6: Evaporating black hole after Page time, when black hole is connected to early Hawking radiation by a wormhole. Alice is near early Hawking radiation and is measuring entanglement between early Hawking radiation and just emitted mode. Bob is infalling into the right black hole and measures entanglement between just emitted mode and it's interior counterpart.

to the AMPS argument. For instance, if black hole was created from the pure state then it has no entangled pair, but at the Page time, the emitted Hawking radiation carries as many degrees of freedom as the remaining black hole, and is maximally entangled with the black hole. Using the duality between maximal entanglement and wormholes, ER = EPR argues that this early Hawking radiation plays the role of the second black hole and is connected to the interior of the first black hole by numerous wormholes. So, the interior mode and early Hawking radiation are not independent and strong subadditivity should not hold true in Equation (2.2.22). Furthermore, it gives a more intuitive interpretation to the loophole in AMPS argument, suggesting that if Alice tries to measure entanglement between early Hawking radiation and just emitted mode, it is equivalent to sending a shock wave through the wormhole, which will result in a firewall for infalling Bob. But if she doesn't attempt to measure it then Bob will see fall through a harmless horizon, thus measuring maximal entanglement between just emitted and interior modes. This equivalence between quantum measurement and creation of shock waves has not been proved for Schwarzchild black holes, but it has been shown that in AdS/CFT small perturbations from maximally entangled thermofield state result in shock wave in the bulk [38]. The ER = EPR conjectures does not resolve the information paradox, but it provides a good amount of evidence that unitarity and equivalence can be combined within the framework of black hole complementarity.

3.5 The ER=EPR Conjecture and Entropy

As we saw, the duality between entanglement and wormholes provides a possible resolution to AMPS argument and existence of firewalls. It also suggests that complementarity successfully combines equivalence principle and unitarity, when the link between entanglement and geometry is taken into account. The ER = EPR conjecture [29] addresses this link, by more general claim that even for an entangled pair of particles, in a quantum theory of gravity there must be a Planckian bridge between them, albeit a very quantum-mechanical bridge which probably cannot be described by classical geometry. There is indirect support for the conjecture. For instance, it is a curious coincidence that despite the nonlocal nature of ER bridges and EPR correlations, they both conspire to prevent superluminal signals. On the ER side, this is a consequence of the topological censorship theorems [39, 40, 41], which forbid traversable wormholes for reasonable energy conditions. Furthermore, motivated by holographic formula (3.3.2) the ER = EPR conjecture also relates the entanglement entropy of black holes connected by an ER bridge to the cross sectional area of the bridge. We assume spacetime is static and work on a constant time slice. Suppose black hole A is joined to some other black holes by an ER bridge. We cut the ER bridge into two pieces, such that one piece contains A and the other piece contains the remaining black holes. Let γ_A be the cut with the smallest cross sectional area and let $a(\gamma_A)$ be its area. Then the entanglement entropy of A is conjectured to be:

$$S_{ER}(A) \equiv a(\gamma_A). \tag{3.5.3}$$

where we ignore any constant coefficients, as they are irrelevant for future discussion. It follows from Equation (3.5.3) that if two black holes A and B are not connected to any other black holes by ER bridges, then $S_{ER}(A) = S_{ER}(B)$. Figure 3-7 gives a 2+1 dimensional example. Similarly, if γ_{AB} is the area minimizing cut that separates

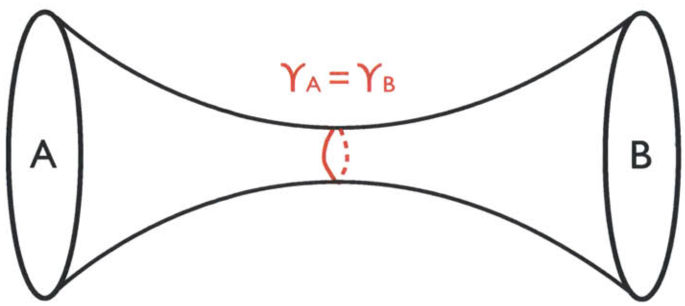

Figure 3-7: A constant time slice of 2+1 dimensional spacetime is a two dimensional surface. We assume the spacetime is static so the slice is well defined. The black hole horizons are circles A and B. They are joined by an ER bridge. The smallest cut separating the ER bridge into a piece containing A and a piece containing B is a closed curve, γ_A. The length of γ_A is $a(\gamma_A) = S_{ER}(A)$. For the simple geometry pictured here, in which A and B are not joined to any other black holes by ER bridges, $S_{ER}(A) = S_{ER}(B)$.

black holes A and B from all other black holes, then:

$$S_{ER}(AB) \equiv a(\gamma_{AB}). \tag{3.5.4}$$

This is zero if A and B are not connected to any other black holes by an ER bridge. In this case the quantum degrees of freedom underlying the joint system AB are conjectured to be in a pure state. Definitions (3.5.3) and (3.5.4) are closely related to the notion of HEE in asymptotically AdS spaces [37, 42] discussed in Section 3.3.

There are two notable differences. First, the minimal surface that defines the HEE of a region A is required to be anchored on the boundary of A, while the cuts through the ER bridge do not meet A in general (see [43] for a further discussion of this difference). Second, in many applications HEE is infinite and must be regularized*, while definitions (3.5.3) and (3.5.4) are finite.

It is not at all apparent from definitions (3.5.3) and (3.5.4) that the entropy so defined obeys the usual inequalities: subadditivity, the triangle inequality, strong subadditivity, and the CLW inequalities. If the inequalities failed, then the ER = EPR conjecture would be wrong. In the next section, we show that the inequalities are satisfied for static, classical ER bridges. This supports the ER = EPR conjecture in this case. Definitions (3.5.3) and (3.5.4) could be extended to non-static spacetimes using an analogue of the covariant holographic entanglement entropy (CHEE) proposal for AdS [43]. We will not consider that extension here, but see [44, 45] for a discussion of the entropy inequalities satisfied by CHEE. It is not possible to check the entropy inequalities for quantum ER bridges as these have yet to be defined (independently of the ER = EPR conjecture).

3.6 Entropy Inequalities

3.6.1 Subadditivity

Consider two black holes, A and B, connected by a static, classical ER bridge (in any spacetime dimension). They may also be connected to other black holes by classical ER bridges. The maximally-extended Schwarzschild and AdS-Schwarzschild metrics are examples involving two black holes. Examples involving more than two black holes have not been constructed analytically but there is no reason they should not exist in principle. Multi-centered black hole solutions have been constructed in supergravity [46], however these black holes are extremal and not connected by ER bridges. We

*A proper subset of the boundary has infinite HEE. An entire connected component of the boundary and the empty set have finite HEE.

claim:

$$S_{ER}(AB) \leq S_{ER}(A) + S_{ER}(B). \tag{3.6.5}$$

This inequality is called subadditivity. Subadditivity is a property of entanglement entropy discussed in Section 1.2.5, so the ER = EPR conjecture would be wrong if equation (3.6.5) failed. By definition $S_{ER}(A) = a(\gamma_A)$, where γ_A is the minimal cut through the ER bridge that divides it into two pieces such that one piece contains A and the other piece contains all other black holes. The cuts γ_B and γ_{AB} are defined similarly. Now the combined cut, $\gamma_A \cup \gamma_B$, separates the ER bridge into a

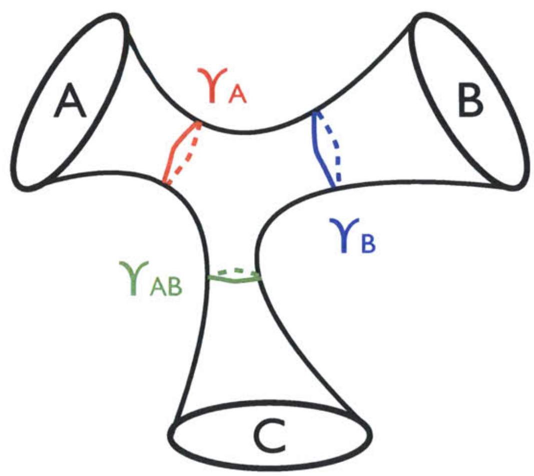

Figure 3-8: An ER bridge on a constant-time slice of 2+1 dimensional spacetime is a two-dimensional surface. (We assume spacetime is static, so the slice is well defined). The black hole horizons are circles A, B, and C, and the minimum area cuts through the ER bridge are closed curves γ_A, γ_B, and γ_{AB}. The ER bridge pictured here has a simple geometry, but the proofs apply to all classical ER bridges.

piece containing AB and a piece containing any other black holes joined to AB by a classical ER bridge. See Figure 3-8 for an example in 2+1 dimensions. It does not generally have the minimum area of such a cut, but it gives an upper bound for $S_{ER}(AB)$:

$$a(\gamma_A \cup \gamma_B) \geq a(\gamma_{AB}) = S_{ER}(AB). \tag{3.6.6}$$

On the other hand, the area of the combined cut is less than or equal to the combined areas of the separate cuts,

$$a(\gamma_A \cup \gamma_B) \leq a(\gamma_A) + a(\gamma_B) = S_{ER}(A) + S_{ER}(B), \tag{3.6.7}$$

where equality is obtained when γ_A and γ_B have no intersection. Equation (3.6.6) and equation (3.6.7) together give subadditivity in equation (3.6.5).

3.6.2 Triangle Inequality

Our next claim is that the ER bridge between A and B satisfies the triangle inequality:

$$|S_{ER}(A) - S_{ER}(B)| \leq S_{ER}(AB). \tag{3.6.8}$$

The triangle inequality is a standard property of entanglement entropy, so equation (3.6.8) is another important check of the ER = EPR conjecture. The combined cut $\gamma_B \cup \gamma_{AB}$ separates the ER bridge into a piece containing A and a piece containing the remaining black holes. (See the 2+1 dimensional example in Figure 3-8.) It does not necessarily have the minimum area of such a cut, but it gives an upper bound for $S_{ER}(A)$:

$$a(\gamma_B \cup \gamma_{AB}) \geq a(\gamma_A) = S_{ER}(A). \tag{3.6.9}$$

The area of the combined cut is less than or equal to the combined areas of the cuts, so

$$a(\gamma_B \cup \gamma_{AB}) \leq a(\gamma_B) + a(\gamma_{AB}) = S_{ER}(B) + S_{ER}(AB), \tag{3.6.10}$$

where equality is obtained if γ_B and γ_{AB} have no intersection. Equation (3.6.9) and 3.6.10 together imply

$$S_{ER}(A) \leq S_{ER}(B) + S_{ER}(AB). \tag{3.6.11}$$

We may assume $S_{ER}(A) > S_{ER}(B)$ without loss of generality and this immediately gives the triangle inequality by equation (3.6.8), as desired.

3.6.3 Strong Subadditivity

The next inequality we consider is strong subadditivity:

$$S_{ER}(A) + S_{ER}(C) \leq S_{ER}(AB) + S_{ER}(BC). \tag{3.6.12}$$

This offers the best support for ER = EPR of the three inequalities we have considered so far, as it is a significantly deeper property of entanglement entropy than either subadditivity or the triangle inequality. Its depth can be seen from the difficulty of proving strong subadditivity on the EPR side for entanglement entropy [47, 48], which was discussed in Section 1.2.6. Strong subadditivity involves three black holes, A, B, and C, and the proof is a more elaborate version of our earlier arguments for subadditivity and the triangle inequality. We consider the area minimizing cuts γ_{AB} and γ_{BC} whose areas are $S_{ER}(AB)$ and $S_{ER}(BC)$. From these we define new cuts, $\tilde{\gamma}_A$ and $\tilde{\gamma}_C$, which split the ER bridge into pieces containing only A and only C, respectively. We show that the total area $a(\tilde{\gamma}_A) + a(\tilde{\gamma}_C)$ is bounded below by $S_{ER}(A) + S_{ER}(C)$ and above by $S_{ER}(AB) + S_{ER}(BC)$. This will suffice to prove strong subadditivity. The key step is defining $\tilde{\gamma}_A$ and $\tilde{\gamma}_C$. The area minimizing cut γ_{AB} splits the ER bridge into two pieces, one containing black holes A and B and the other containing all remaining black holes. Let these two pieces be ab and cd. Similarly, let γ_{BC} split the ER bridge into two pieces bc and ad. The combinations we need are

$$\tilde{\gamma}_A \equiv \gamma_{AB}|_{ad} \cup \gamma_{BC}|_{ab}, \tag{3.6.13}$$

$$\tilde{\gamma}_C \equiv \gamma_{AB}|_{bc} \cup \gamma_{BC}|_{cd}, \tag{3.6.14}$$

where $\gamma_{AB}|_{ad}$ is the restriction of γ_{AB} to ad. The new cut $\tilde{\gamma}_A$ splits the ER bridge into a piece containing black hole A and a piece containing all remaining black holes, while $\tilde{\gamma}_C$ does the same with respect to black hole C. Figure 3-9 gives an example in 2+1 dimensions, involving four black holes connected by a two-dimensional ER bridge. For all classical ER bridges in all dimensions, the total area $a(\tilde{\gamma}_A) + a(\tilde{\gamma}_C)$ is

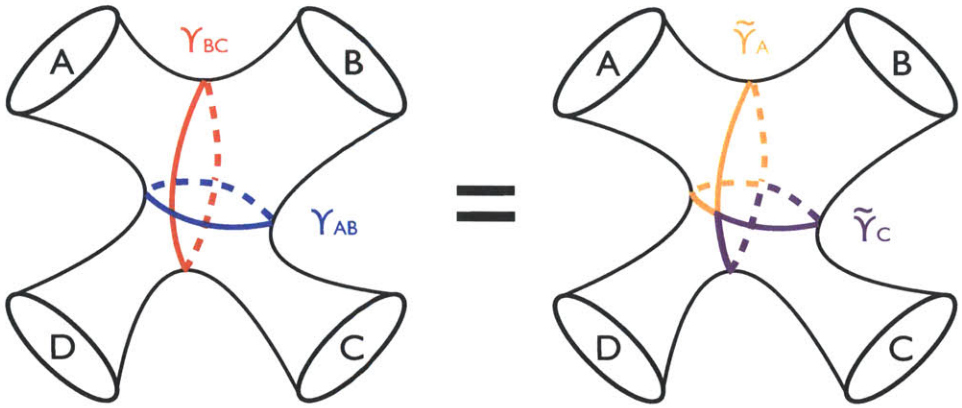

Figure 3-9: As in Figure 3-8, except now with four black holes, A, B, C, and D. Starting from the cuts γ_{AB} and γ_{BC}, we construct new cuts $\tilde{\gamma}_A$ and $\tilde{\gamma}_C$ (equations 3.6.13-3.6.14). Strong subadditivity is proved by bounding the sizes of $\tilde{\gamma}_A$ and $\tilde{\gamma}_C$.

bounded below by the total area of the minimizing cuts:

$$a(\tilde{\gamma}_A) + a(\tilde{\gamma}_C) \geq a(\gamma_A) + a(\gamma_C) = S_{ER}(A) + S_{ER}(C). \tag{3.6.15}$$

We also have the upper bound:

$$\begin{aligned} a(\tilde{\gamma}_A) + a(\tilde{\gamma}_C) &\leq a(\gamma_{AB}|_{ad}) + a(\gamma_{BC}|_{ab}) + a(\gamma_{AB}|_{bc}) + a(\gamma_{BC}|_{cd}) \\ &= S_{ER}(AB) + S_{ER}(BC). \end{aligned} \tag{3.6.16}$$

Combining equations 3.6.15 and 3.6.16 gives strong subadditivity, equation 3.6.12, as desired.

3.6.4 Monogamy

Finally, we will show that three black holes A, B, and C, connected by a static, classical ER bridge satisfy:

$$\begin{aligned} &S_{ER}(A) + S_{ER}(B) + S_{ER}(C) + S_{ER}(ABC) \\ &\quad \leq S_{ER}(AB) + S_{ER}(BC) + S_{ER}(AC). \end{aligned} \tag{3.6.17}$$

Systems obeying this inequality are called monogamous. Not all entangled systems are monogamous in this sense, so Equation (3.6.17) restricts the set of states that can describe static, classical ER bridges. We return to the implications of this fact for ER = EPR in section 3.7. Let D be all other black holes connected to A, B, and C by classical ER bridges (D is not necessarily a single black hole). We define:

$$\begin{aligned}
\tilde{\gamma}_A &= \gamma_{AB}|_{ac\cap ad} \cup \gamma_{AC}|_{ad\cap ab} \cup \gamma_{AD}|_{ab\cap ac}, &(3.6.18)\\
\tilde{\gamma}_B &= \gamma_{BA}|_{bc\cap bd} \cup \gamma_{BC}|_{bd\cap ba} \cup \gamma_{BD}|_{ba\cap bc}, &(3.6.19)\\
\tilde{\gamma}_C &= \gamma_{CA}|_{cb\cap cd} \cup \gamma_{CB}|_{cd\cap ca} \cup \gamma_{CD}|_{ca\cap cb}, &(3.6.20)\\
\tilde{\gamma}_D &= \gamma_{DA}|_{db\cap dc} \cup \gamma_{DB}|_{dc\cap da} \cup \gamma_{DC}|_{da\cap db}. &(3.6.21)
\end{aligned}$$

$\tilde{\gamma}_A$ is a combination of the area-minimizing cuts γ_{AB}, γ_{AC}, and γ_{AD}, such that each of the three area-minimizing cuts is restricted to the regions defined by the other two. So $\tilde{\gamma}_A$ separates the ER bridge into a piece containing A and a piece containing all other black holes. $\tilde{\gamma}_B$, $\tilde{\gamma}_C$, and $\tilde{\gamma}_D$ are defined similarly. See Figure 3-10 for an example involving four 2+1 dimensional black holes connected by a two-dimensional ER bridge. The total area of the new cuts is bounded below by:

$$a(\tilde{\gamma}_A) + a(\tilde{\gamma}_B) + a(\tilde{\gamma}_C) + a(\tilde{\gamma}_D) \geq S_{ER}(A) + S_{ER}(B) + S_{ER}(C) + S_{ER}(D). \quad (3.6.22)$$

and bounded above by:

$$a(\tilde{\gamma}_A) + a(\tilde{\gamma}_B) + a(\tilde{\gamma}_C) + a(\tilde{\gamma}_D) \leq S_{ER}(AB) + S_{ER}(BC) + S_{ER}(AC), \quad (3.6.23)$$

Combining bounds 3.6.22 and 3.6.23, and using $S_{ER}(D) = S_{ER}(ABC)$, we obtain monogamy, Equation (3.6.17).

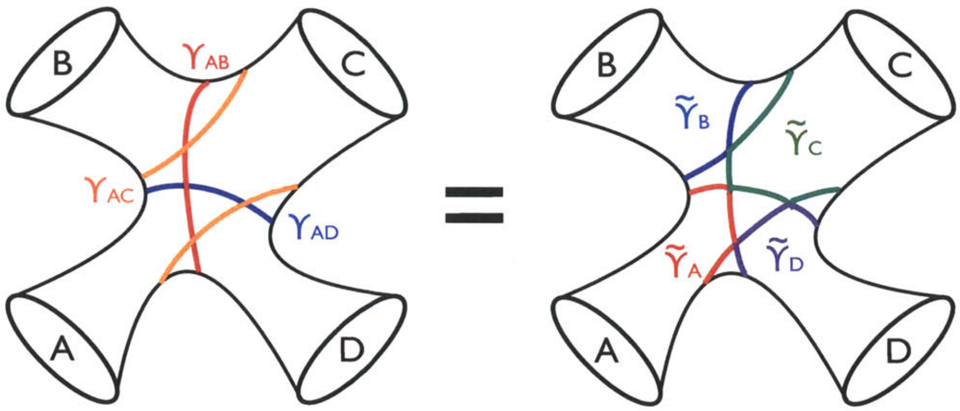

Figure 3-10: A constant time slice of 2+1 dimensional spacetime is a two dimensional surface. Black holes A, B, C, and D (black circles) are connected by a two dimensional ER bridge. Cuts through the bridge are rearranged to give new cuts, $\tilde{\gamma}_A$, $\tilde{\gamma}_B$, $\tilde{\gamma}_C$, and $\tilde{\gamma}_D$, each of which cuts out a single black hole (see equations 3.6.18-3.6.21). Monogamy is proved by bounding the sizes of the cuts.

3.6.5 Further Inequalities

Combining equation (3.6.12) and the triangle inequality (3.6.8) gives an alternate version of strong subadditivity:

$$S_{ER}(ABC) + S_{ER}(B) \leq S_{ER}(AB) + S_{ER}(BC). \tag{3.6.24}$$

In quantum information theory, it is sometimes convenient to work with the mutual information,

$$I_{ER}(A:B) \equiv S_{ER}(A) + S_{ER}(B) - S_{ER}(AB), \tag{3.6.25}$$

the conditional mutual information,

$$I_{ER}(A:C|B) \equiv S_{ER}(AB) + S_{ER}(BC) - S_{ER}(B) - S_{ER}(ABC) \geq 0, \tag{3.6.26}$$

and the tripartite information [49, 8, 35],

$$\begin{aligned} I^3_{ER}(A:B:C) &\equiv S_{ER}(A) + S_{ER}(B) + S_{ER}(C) \qquad (3.6.27) \\ &- S_{ER}(AB) - S_{ER}(BC) - S_{ER}(AC) + S_{ER}(ABC). \end{aligned}$$

With these definitions, subadditivity becomes $I_{ER}(A:B) \geq 0$, strong subadditivity becomes $I_{ER}(A:C|B) \geq 0$, and monogamy becomes $I^3_{ER}(A:B:C) \leq 0$. Cadney, Linden, and Winter (CLW) [33, 34] have obtained a new set of inequalities for entanglement entropy. Strong subadditivity and monogamy together imply the CLW inequalities [50].

3.7 GHZ State

The entropy inequalities proved in the previous section apply to static, classical ER bridges. The geometrical definition of entanglement entropy given by equations (3.5.3)-(3.5.4) does not apply to non-static spacetimes. However, it could be generalized by following the example of CHEE for non-static, asymptotically AdS spacetimes [43]. CHEE appears to satisfy the same entropy inequalities as HEE [44, 45], so we expect that all (not necessarily static) classical ER bridges satisfy the entropy inequalities considered in section 3.6. Quantum ER bridges have yet to be defined independently of the ER = EPR conjecture. If the conjecture is correct, quantum ER bridges should satisfy subadditivity, the triangle inequality, strong subadditivity, and the CLW inequalities, as these are general properties of entanglement entropy. However, unlike classical ER bridges, quantum ER bridges cannot be monogamous, because there are quantum states with positive tripartite information. A simple example is the four qubit pure state $|GHZ_4\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}$ [51, 52], which has $I^3(A:B:C) = +1$. If there is a quantum ER bridge corresponding to this state, it must be very unlike any classical ER bridge. This shows that the ER bridge depends not just on the entanglement entropy, but on the pattern of entanglement. Suppose we have N copies of $|GHZ_4\rangle$. Let the qubits of the ith copy be A_i, B_i, C_i, and D_i. We could collapse all of the A_i into a black hole A, collapse the B_i into a black hole B, and so on. If the number of qubits is large, then the black holes are macroscopic and have arbitrarily large entanglement entropy. However, the tripartite information of the black holes remains positive, so there is no classical ER bridge describing their entanglement. This does not necessarily contradict ER = EPR because it is still possible for the entanglement between the black holes to be described by a quantum ER bridge. This possibility cannot be ruled out without a better understanding of quantum ER bridges. It is striking that quantum bridges do not necessarily become classical in the limit of infinite entropy. Classical ER bridges seem to emerge when quantum correlations dominate classical correlations. For example, Bell pairs contain purely quantum correlations. We expect large black holes built out of Bell pairs have classical ER bridges, because all combinations of Bell pairs are monogamous. However, the $|GHZ_4\rangle$ state contains a mix of classical and quantum correlations. As we increase the number of $|GHZ_4\rangle$ states, the ratio of quantum to classical correlations stays fixed, the tripartite information stays positive, and a classical geometry fails to emerge.

3.8 Discussion

The ER = EPR conjecture is part of much broader research dedicated to the investigation of the link between geometry and quantum information theory. This link has been mainly studied in the context of dualities between field theory living in the boundary and the gravity living in the bulk. One of the main directions is the study of the correspondence between entanglement and wormholes. This connection was first recognized in AdS/CFT correspondence where wormhole between two asymptotically AdS regions is dual to two non-interacting conformal field theories in a thermally entangled state [36]. It was also noticed that the area of the minimal surface cut represent the entanglement entropy and the length of the wormhole is proportional to correlations between two dual CFT's. This idea was generalized in [53], proposing that spacetime connectedness is AdS is related to quantum entanglement in the dual field theory. Further, ER = EPR conjectured that entanglement is equivalent to the existence of wormholes in spacetime [29]. This is more general proposal, suggesting that entanglement should be identified with the existence of a wormhole. However, the example of the GHZ state from Section 3.7 [32] suggests that arbitrary entangled state can not be represented as a classical Einstein-Rosen bridge and further classification is necessary. Moreover, authors of [54] and [55] argued that thermal state is very special state and not every entangled states can be represented as a wormhole. They used the eigenvalue thermalization hypothesis (ETH) and random matrix method to argue that local correlations in a typical entangled state are weak and do not correspond to a semiclassical wormhole in the bulk. This controversy suggest that the relation between entanglement and wormholes is more complex and concrete mathematical framework is necessary to convince the skeptic that this link is universal (if it is universal). Another important direction in geometry-quantum research is related to emergence of the spacetime in the quantum theory. If one assumes that the total Hamiltonian $\mathbf{H}$ and the total density matrix ρ fully specify our physical world [56], one would expect the 3D space and semiclassical world to emerge from nothing more than two Hermitian matrices [57]. Particular manifestation of this idea is presented in [58], where author argues that tensor network representation of the entangled ground state can be reinterpreted as a kind of skeleton for an emergent holographic space, showing that for the large N limit this leads to the emergence of smooth AdS spacetime. Another example of this is dubbed exact holographic mapping [59], which suggests that geometry in the bulk of the many-body boundary is nothing but the boundary theory viewed in a different basis. It is shown that the space-time geometry is determined by the structure of correlations and quantum entanglement of the many-body quantum state.

In the search of fundamental laws of nature, it is natural to think about the relationship between geometry and quantum theory. Space is a specific factorization of the global Hilbert space and one would expect a natural law to exist to predict this particular factorization. It is indeed possible that working theory of quantum gravity is necessary for this emergence [60], but science is about speculations and optimism. And physicist's job is to use known laws of nature to make predictions and postulate new theories to resolve paradoxes. Study of the link between geometry and quantum information is an exciting area of research and there is still a lot to be understood.

Chapter 4

Sharpening the Second Law of Thermodynamics

More than a century after its formulation, the second law of thermodynamics remains at the forefront of physics research, with continuing progress on generalizing its applications and clarifying its foundations. For example, it is being extended to non-equilibrium statistical mechanics [61], quantum heat engines [62], biological self-replication [63] and cosmological inflation [64]. As to the quantum-mechanical version of the second-law, Seth Lloyd showed that can be derived from imperfectly known quantum evolution [65], and one of us showed how it can be generalized to observed open systems [66, 64]. The goal of this chapter is to complete this generalization with an assumption of unitary cosmology and by providing the required mathematical proofs.

4.1 Object, Subject and Environment

As emphasized by von Neumann [4] and Feynman [56], state of a quantum system is completely described by a density matrix ρ, which encodes everything we need to know to make the best possible predictions for its future behavior. However, if you are interested in using physics to make predictions about your own future, knowing ρ for the entire universe (or multiverse) is neither sufficient nor necessary [64].

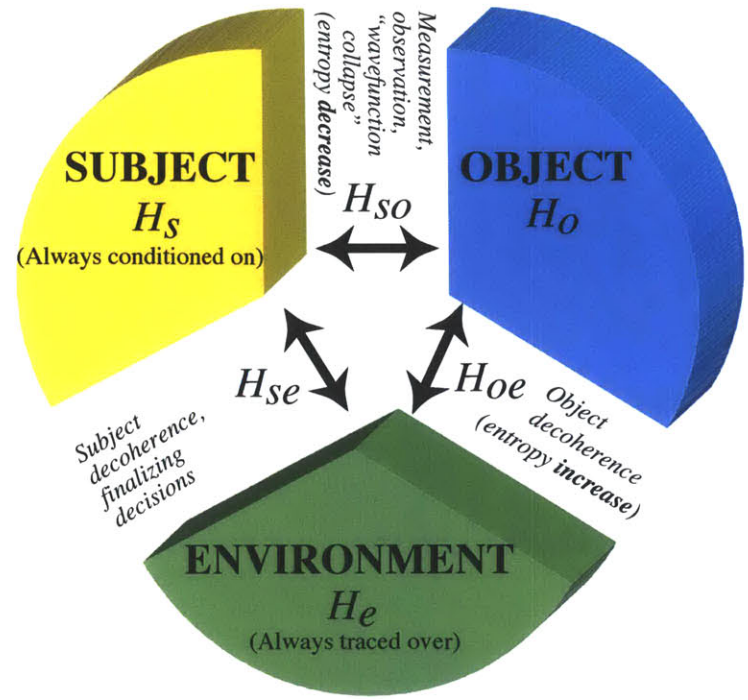

Figure 4-1: "EOS-decomposition" of the universe. The subsystem Hamiltonians H_s, H_o, H_e and the interaction Hamiltonians H_so, H_oe, H_se can cause qualitatively different effects, providing a unified picture including both observation and decoherence.

1. **Is not sufficient:** You also need to know what branch of the global wavefunction you are in. In particular, you need to take into account what you know about your location, both in 3D space and in Hilbert space.
2. **Is not necessary:** You only need to know the quantum state "nearby", both in 3D space and in Hilbert space.

To predict what you will observe in a quantum physics lab, you need to take into account both which of the many existing physics labs you happen to be in and also which quantum state preparation has been performed. On the other hand, you do not need to take into account the current state of the Andromeda Galaxy or branches of the wavefunction that have permanently decohered from your own.

Table 4.1: Summary of the two basic quantum processes discussed in the text.

For observer to predict future, global ρ...	...is not sufficient	...is not necessary
Mathematical Operation	Condition (on subject degrees of freedom)	Marginalize (over environment degrees of freedom)
Interaction	Object-Subject	Object-Environment
Process	Observation	Decoherence
Dynamics	$\rho_{ij} \mapsto \rho_{ij}^{(k)} = \rho_{ij}\frac{S_{ik}S_{jk}^*}{p_k}$	$\rho_{ij} \mapsto \rho_{ij}E_{ij}$
Entropy Inequality	Decrease: $\sum_k p_k S\left(\rho^{(k)}\right) \leq S(\rho)$	Increase: $S(\rho) \leq S\left(\rho \circ E\right)$

To address this issues you can always decompose the total system (the entire universe) into three subsystems as illustrated in Figure 4-1:

1. The **subject**, consisting of the degrees of freedom associated with the subjective perceptions of the observer.

2. The **object**, consisting of the degrees of freedom that the observer is studying, e.g., polarization of the photon.

3. The **environment**, consisting of everything else,*i.e.*, all the environmental degrees of freedom that the observer is not aware of.

This EOS decomposition of the universe allows a corresponding decomposition of the Hamiltonian:

$$H = H_s + H_o + H_e + H_{\rm so} + H_{\rm se} + H_{\rm oe} + H_{\rm soe}, \tag{4.1.1}$$

where the first three terms act on individual systems and the second three terms represent pairwise interactions between subsystems, and the third term represents any irreducible three-way interaction. In this picture, one can study the dynamics of the object as a combination of it's internal dynamics (given by $H_{\rm o}$) and it's interaction with subject and environment. The H_{so} involves quantum measurement and H_{oe} produces decoherence, which selects certain basis and makes the object act classically. Thus, computing the correct density matrix for your object of interest therefore involves two steps [64]:

1. **Condition** on what you know (on all subject degrees of freedom). Which is due to the fact that subject perceives only single basis state.
2. **Marginalize** over what you don't care about (partial-trace over all environment degrees of freedom).

The first step, including quantum state preparation and observation, can be thought of as the quantum generalization of Bayes' Theorem [64]. The second step produces decoherence, helping explain the emergence of a classical world [67, 68, 69, 70]. As we will see below, decoherence always increases entropy whereas observation on average decreases it. Although the latter was proved by Shannon for the case of classical physics, the corresponding quantum theorem has hitherto eluded proof. We will sharpen the second law as follows:

1. **Observation:** When an object is probed by the subject, its entropy on average decreases.
2. **Decoherence:** When an object is probed by the environment, its entropy increases.

Below we will define "probed" and provide rigorous mathematical proofs of these entropy inequalities.

4.2 Probing: Decoherence and Measurement

Let us begin by briefly reviewing the formalism of [66, 64] necessary for our proofs. We define *probing* as a nontrivial interaction between the object and some other system that leaves the object unchanged in some basis, *i.e.*, such that the unitary dynamics $\mathbf{U}$ merely changes the system state in a way that depends on the object state $|o_i\rangle$.

4.2.1 Object-Environment

If the object is probed by the environment the unitary dynamics of object-environment system is given by

$$\mathbf{U}|e_0\rangle|o_i\rangle = |\epsilon_i\rangle|o_i\rangle. \tag{4.2.2}$$

where $|e_0\rangle$ and $|\epsilon_i\rangle$ denote the initial and final states of the environment for the object state $|o_i\rangle$. Thus, the initial density matrix $\rho_{oe} = |e_0\rangle\langle e_0|\otimes\rho$ of the object-environment system, where $\rho = \sum_{ij}\langle o_i|\rho|o_j\rangle|o_i\rangle\langle o_j|$, will evolve as:

$$\begin{aligned}
\rho_{oe} &\mapsto U\rho\mathbf{U}^\dagger = U|e_0\rangle\langle e_0|\rho\mathbf{U}^\dagger \\
&= \sum_{ij}\langle o_i|\rho|o_j\rangle U|e_0\rangle|o_i\rangle\langle e_0|\langle o_j|\mathbf{U}^\dagger \\
&= \sum_{ij}\langle o_i|\rho|o_j\rangle|\epsilon_i\rangle|o_i\rangle\langle\epsilon_j|\langle o_j|.
\end{aligned} \tag{4.2.3}$$

Tracing over the environment, we see that the reduced density matrix for the object evolves as

$$\begin{aligned}
\rho &\mapsto \mathrm{tr}_{\,e}\rho_{oe} \equiv \sum_k\langle e_k|\rho_{oe}|e_k\rangle \\
&= \sum_{ijk}\langle o_i|\rho|o_j\rangle\langle\epsilon_j|e_k\rangle\langle e_k|\epsilon_i\rangle|o_i\rangle\langle o_j| \\
&= \sum_{ij}\langle o_i|\rho|o_j\rangle\langle\epsilon_j|\epsilon_i\rangle|o_i\rangle\langle o_j|,
\end{aligned} \tag{4.2.4}$$

where we used the fact that $\sum_k|e_k\rangle\langle e_k| = I$. If the matrix $\mathbf{E}$ is defined as $E_{ij} \equiv \langle\epsilon_j|\epsilon_i\rangle$ then the evolution of the object density matrix is given by:

$$\rho \mapsto \rho\circ\mathbf{E}, \tag{4.2.5}$$

where the symbol $\circ$ denotes what mathematicians know as the *Schur product.* Schur multiplying two matrices simply corresponds to multiplying their corresponding components, *i.e.*, $(\rho\circ\mathbf{E})_{ij} = \rho_{ij}E_{ij}$.

4.2.2 Object-Subject

If the object instead is probed by an observer the unitary dynamics of the object-subject system is given by:

$$\mathbf{U}|s_0\rangle|o_i\rangle = |\sigma_i\rangle|o_i\rangle. \tag{4.2.6}$$

where $|s_0\rangle$ and $|\sigma_i\rangle$ denote the initial and final states of the subject for the object state $|o_i\rangle$. Thus, the initial density matrix $\rho_{os} = |s_0\rangle\langle s_0|\otimes\rho$ of the object-environment system, where $\rho = \sum_{ij}\langle o_i|\rho|o_j\rangle|o_i\rangle\langle o_j|$, will evolve as:

$$\begin{aligned}\rho_{os} &\mapsto U\rho_{os}\mathbf{U}^\dagger = U|s_0\rangle\langle s_0|\rho\mathbf{U}^\dagger \\ &= \sum_{ij}\langle o_i|\rho|o_j\rangle U|s_0\rangle|o_i\rangle\langle s_0|\langle o_j|\mathbf{U}^\dagger \\ &= \sum_{ij}\langle o_i|\rho_o|o_j\rangle|\sigma_i\rangle|o_i\rangle\langle\sigma_j|\langle o_j|. \end{aligned} \tag{4.2.7}$$

Let $|s_k\rangle$ denote the basis states that the subject can perceive, which are robust to decoherence and will correspond to "pointer states" [71] for the case of human observer. Since the subject will rapidly decohere, we can use the decoherence formula from previous section with projections $P_k = |s_k\rangle\langle s_k|$ into this basis, which gives:

$$\begin{aligned}\rho_{os} &\mapsto \sum_k P_k\rho_{os}P_k = \sum_k |s_k\rangle\langle s_k|\rho_{os}|s_k\rangle\langle s_k| \\ &= \sum_{ijk}\langle o_i|\rho|o_j\rangle\langle s_k|\sigma_i\rangle\langle\sigma_j|s_k\rangle|s_k\rangle\langle s_k|\otimes|o_i\rangle\langle o_j| \\ &= \sum_k |s_k\rangle\langle s_k|\otimes\tilde{\rho}^{(k)}, \end{aligned} \tag{4.2.8}$$

where we denote by:

$$\tilde{\rho}^{(k)} \equiv \sum_{ij}\langle o_i|\rho|o_j\rangle\langle s_k|\sigma_i\rangle\langle\sigma_j|s_k\rangle|o_i\rangle\langle o_j| \tag{4.2.9}$$

Thus, conditioned on the subject perception state $|s_k\rangle$, the normalized state for the object is given by:

$$\rho^{(k)} = \frac{\tilde{\rho}^{(k)}}{\operatorname{tr}\tilde{\rho}^{(k)}} \tag{4.2.10}$$

where subject perceives $|s_k\rangle$ with probability $p_k = \mathrm{tr}\,\tilde{\rho}^{(k)}$. If we denote by the vector $\mathbf{s}^k$ the k^{th} column of the matrix $S_{ik} \equiv \langle s_k|\sigma_i\rangle$, *i.e.*, $s_i^k \equiv S_{ik}$ and $\rho_{ij}^{(k)} = \rho_{ij}S_{ik}S_{jk}^*/p_k$. We can write the final object state and probability as:

$$\rho^{(k)} = \frac{\rho \circ (\mathbf{s}^k\mathbf{s}^{k\dagger})}{p_k}, \quad p_k = \sum_i \rho_{ii}|s_i^k|^2. \tag{4.2.11}$$

We can think of this as the quantum-mechanical version of Bayes' Theorem [64], showing how our state of knowledge about a system gets updated by conditioning on new observed information. These effects of decoherence and observation are intimately related. Since the states $|s_k\rangle$ form a basis, let's define $F_{ij} = \langle\sigma_j|\sigma_i\rangle = \sum_k \langle\sigma_j|s_k\rangle\langle s_k|\sigma_i\rangle = \sum_k S_{jk}^* S_{ik}$, so

$$\mathbf{F} = \mathbf{S}\mathbf{S}^\dagger. \tag{4.2.12}$$

and from the equations (4.2.11) and (4.2.12) we conclude that

$$\rho \circ \mathbf{F} = \sum_k p_k \rho^{(k)}. \tag{4.2.13}$$

This means that decoherence (eq. 4.2.5) can be interpreted as an observation that we do not know the outcome of: after being probed by its environment (setting $\mathbf{F} = \mathbf{E}$), the object is in one of the states $\rho^{(k)}$ (with probability p_k), and we simply do not know which. The most revealing probing is when $\mathbf{S} = \mathbf{E} = \mathbf{I}$, the identity matrix: then the system gains complete information about the object, so decoherence makes ρ diagonal (so-called von Neumann reduction [4]) and observation produces pure states $\rho^{(k)} = |\tilde{o}_k\rangle\langle\tilde{o}_k|$, for some orthonormal basis $|\tilde{o}_k\rangle$. Another interesting special case is when all elements S_{ij} are zero or unity. Then the decoherence equation (4.2.5) reduces to what is known as Lüders projection [72]

$$\rho \circ \mathbf{E} = \sum_k \mathbf{P}_k \rho \mathbf{P}_k, \tag{4.2.14}$$

and the observation equation (4.2.11) gives $\rho^{(k)} \propto \mathbf{P}_k \rho \mathbf{P}_k$, where $\mathbf{P}_k \equiv \sum_i S_{ik} |o_i\rangle\langle o_i|$ are orthogonal projection operators (satisfying $\mathbf{P}_i \mathbf{P}_j = \delta_{ij} \mathbf{P}_i$, $\sum \mathbf{P}_i = \mathbf{I}$). The least revealing probing is the trivial case when $\rho^{(k)}$ and $\rho \circ \mathbf{E}$ equal ρ up to a unitary transformation, so that the subject or environment learns nothing about the object. We define "probing" to exclude this trivial case, which occurs for example when $\mathbf{S}$ is of the form $S_{jk} = e^{i(\theta_j + \phi_k)}$ or, for the decoherence case, when ρ is diagonal.

4.3 Entropy Inequalities (S-Theorems)

We will now prove the main result of this paper: that observation on average decreases entropy, while decoherence increases entropy. Specifically, we will prove the theorem

$$\sum_k p_k S\left(\rho^{(k)}\right) \leq S(\rho) \leq S\left(\sum_k p_k \rho^{(k)}\right), \tag{4.3.15}$$

relating the expected entropy after observation (left), the initial entropy (middle) and the entropy after decoherence (right). Both "$\leq$" become "$<$" when the probing is nontrivial. We will refer to these entropy inequalities as two S-theorems. Our proof below holds for the general types of observation from equation (4.2.11) and decoherence from equation (4.2.5), and for a very general definition of entropy: for any quantify of the form

$$S(\rho) \equiv \operatorname{tr} h(\rho) \tag{4.3.16}$$

where h is a concave function on the unit interval ($h''(x) < 0$ for $0 \leq x \leq 1$). This includes the Shannon/von Neumann entropy ($h(x) = -x \ln x$), the linear entropy ($h(x) = 1 - x^2$), the rescaled exponentiated Renyii entropy ($H(x) = \pm x^\alpha$) and the log-determinant ($h(x) = \ln x$).

In Appendix A.1, we study interactions that are more general than probing, obtaining the results illustrated in Figure 4-2. By providing counterexamples that violate both inequalities in equation (4.3.15), we prove that for a general Positive Operator Valued Measure (POVM) [73], neither of the two S-theorems holds true. We also show that our proof of the left (observation) inequality of equation (4.3.15) can be

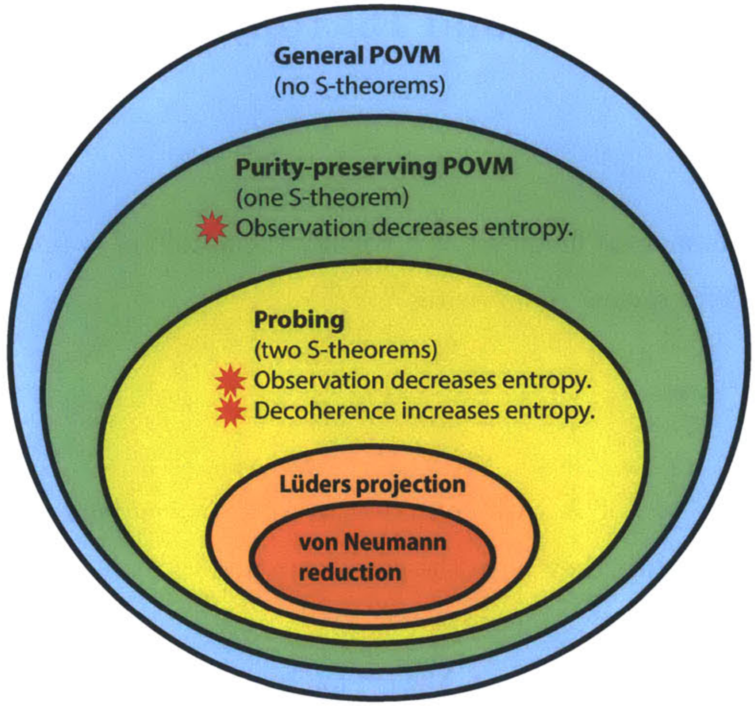

Figure 4-2: Classification of the most general quantum measurements(POVM) and the S-theorems that they satisfy.

generalized from probing to what we term a purity-preserving POVM, "PPPOVM", the special type of POVM that maps any pure state into pure states. We find that for PPPOVMs, observation on average decreases entropy, but decoherence does *not* always increase entropy. The interactions we defined as probing are simply the subset of PPPOVMs that leave the object unchanged in at least one basis. These are the most general interactions in Figure 4-2 that can be interpreted as measurements. To see this, consider that simply replacing the object state by some fixed and *a priori* known pure state (as in counterexample 2 in Appendix A.1) is a PPPOVM, and it would be ridiculous to view this as a measurement of the object.

4.4 Majorization and Entropy

Our proof uses numerous inequalities involving the notion of *majorization* [74], which we will now briefly review. One writes

$$\boldsymbol{\lambda} \succ \boldsymbol{\mu} \tag{4.4.17}$$

and says that the vector $\boldsymbol{\lambda}$ with components λ_1, ...,λ_n *majorizes* the vector $\boldsymbol{\mu}$ with components μ_1, ...,μ_n if they have the same sum and

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i \quad \text{for } j = 1, \ldots, n, \tag{4.4.18}$$

i.e., if the partial sums of the latter never beat the former: $\lambda_1 \geq \mu_1$, $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$, *etc.* It is not difficult to show (see, *e.g.*, [74] or Appendix A of [64]) that if $\boldsymbol{\lambda} \succ \boldsymbol{\mu}$, then

$$\sum_i h(\lambda_i) \leq \sum_i h(\mu_i) \tag{4.4.19}$$

for any concave function h. This means that if the vectors are probability distributions (so that $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$) with entropies defined as

$$S(\boldsymbol{\lambda}) = \sum_i h(\lambda_i), \tag{4.4.20}$$

then the majorization $\boldsymbol{\lambda} \succ \boldsymbol{\mu}$ implies the entropy inequality $S(\boldsymbol{\lambda}) \leq S(\boldsymbol{\mu})$. Letting $\boldsymbol{\lambda}(\rho)$ denote the eigenvalues of the density matrix ρ sorted in decreasing order, and using equation (4.3.16), we thus have the powerful result

$$\boldsymbol{\lambda}(\rho_1) \succ \boldsymbol{\lambda}(\rho_2) \implies S(\rho_1) \leq S(\rho_2), \tag{4.4.21}$$

so if two density matrices ρ_1 and ρ_2 satisfy $\boldsymbol{\lambda}(\rho_1) \succ \boldsymbol{\lambda}(\rho_2)$, then they satisfy the entropy inequality $S(\rho_1) \leq S(\rho_2)$.

4.5 The Proof

The right part of the entropy theorem (4.3.15) (that decoherence increases entropy) was proven in [64]. By equation (4.2.13), it is equivalent to

$$S(\rho \circ \mathbf{E}) \geq S(\rho), \tag{4.5.22}$$

which follows from (4.4.21) and the majorization

$$\boldsymbol{\lambda}(\rho \circ \mathbf{E}) \prec \boldsymbol{\lambda}(\rho), \tag{4.5.23}$$

which is Corollary J.2.a in [74] (their equation 7), which in turn follows from a 1985 theorem by Bapat and Sunder. To prove the other half of the entropy theorem (4.3.15) (that observation decreases entropy), we will need the following theorem, which is proven in Appendix A: For any Hermitean matrix ρ and any complete orthogonal set of projection operators $\mathbf{P}_i$ (satisfying $\sum_i \mathbf{P}_i = \mathbf{I}$, $\mathbf{P}_i\mathbf{P}_j = \delta_{ij}\mathbf{P}_i$, $\mathbf{P}_i^\dagger = \mathbf{P}_i$),

$$\sum_i \boldsymbol{\lambda}(\mathbf{P}_i\rho\mathbf{P}_i) \succ \boldsymbol{\lambda}(\rho) \succ \boldsymbol{\lambda}\left(\sum_i \mathbf{P}_i\rho\mathbf{P}_i\right). \tag{4.5.24}$$

Applying (4.4.21) to the left part gives

$$\begin{aligned}
S(\rho) &\geq S\left(\sum_k \boldsymbol{\lambda}(\mathbf{P}_k\rho\mathbf{P}_k)\right) = S\left(\sum_k p_k \boldsymbol{\lambda}\left(\frac{\mathbf{P}_k\rho\mathbf{P}_k}{p_k}\right)\right) = \\
&= \sum_i h\left(\sum_k p_k \boldsymbol{\lambda}_i\left(\frac{\mathbf{P}_k\rho\mathbf{P}_k}{p_k}\right)\right) \geq \\
&\geq \sum_{ik} p_k h\left(\boldsymbol{\lambda}_i\left(\frac{\mathbf{P}_k\rho\mathbf{P}_k}{p_k}\right)\right) = \sum_k p_k S\left(\frac{\mathbf{P}_k\rho\mathbf{P}_k}{p_k}\right),
\end{aligned}$$

where we used Jensen's inequality in the penultimate step. This, (4.5.24) and (4.4.21) thus imply that for any density matrix ρ,

$$\sum_i p_i S\left(\frac{\mathbf{P}_i\rho\mathbf{P}_i}{p_i}\right) \leq S(\rho) \leq S\left(\sum_i \mathbf{P}_i\rho\mathbf{P}_i\right), \tag{4.5.25}$$

where $p_i \equiv \mathrm{tr}\,\mathbf{P}_i\rho\mathbf{P}_i$. The right half of this double inequality is an alternative proof of equation (4.5.22) for the special case where decoherence is a Lüders projection as in equation (4.2.14). Using instead the left half of (4.5.25), we obtain the following proof of the remaining (left) part of our entropy theorem (4.3.15):

$$\begin{aligned}
\langle S\rangle &\equiv \sum_k p_k S(\rho^{(k)}) = \sum_k p_k S(\rho^{(k)} \otimes |s_k\rangle\langle s_k|) = \\
&= \sum_k p_k S\left(\frac{\mathbf{P}_k\mathbf{U}\rho^*\mathbf{U}\mathbf{P}_k}{p_k}\right) \leq S(\mathbf{U}\rho^*\mathbf{U}) = \\
&= S(\rho^*) = S(\rho\otimes|s_*\rangle\langle s_*|) = S(\rho).
\end{aligned} \tag{4.5.26}$$

Here $\rho^* \equiv \rho\otimes|s_*\rangle\langle s_*|$ is the initial state of the combined object-observer system, which the observation process evolves into $\mathbf{U}\rho^*\mathbf{U}$, which in turn decoheres into $\sum_k \mathbf{P}_k\mathbf{U}\rho^*\mathbf{U}\mathbf{P}_k = \sum_k p_k\rho^{(k)}\otimes|s_k\rangle\langle s_k|)$ as derived in [64], where $\mathbf{P}_k \equiv \mathbf{I}\otimes|s_k\rangle\langle s_k|$ are projection operators acting on the combined object-observer system. The first and last equal signs in equation (4.5.26) hinge on the fact that tensor multiplying a density matrix by a pure state leaves its entropy unchanged, merely augmenting its spectrum by a number of vanishing eigenvalues. The inequality step uses (4.5.25), and the subsequent step uses the fact that unitary evolution leaves entropy unchanged.

4.6 Discussion

We have proved the entropy inequality (4.3.15), which states that decoherence increases entropy whereas observation on average decreases it. Its left half is a direct generalization of the Groenewold-Lindblad inequality [75], which corresponds to the special case of the projective (Lüders) form of measurement and the special case of von Neumann entropy; our results also hold Renyii entropy, linear entropy and indeed any entropy defined by a concave function. Both of these entropy inequalities hold for interactions *probing* the object, defined as the most general interactions leaving the object unchanged in some basis. Of the various classes of interactions we considered, probing constitutes the most general one that can be interpreted as a measurement of the object. We showed that our observation inequality, but not our decoherence inequality, holds also for more general interactions that are purity-reserving POVMs, generalizing Ozawa's result for quasi-complete measurements beyond von Neumann entropy [76]. None of our entropy inequalities hold for arbitrary POVMs.

To prove inequalities (4.3.15), we used the link between unitary quantum mechanics and spectral majorization. This link was first noticed by Uhlmann [77] and later proved extremely helpful in the study of pure state transformations and entanglement [78]. We showed that majorization techniques can also be used to provide simple alternative proofs of entropy inequalities for observation and decoherence. Also it complements Holevo's inequality [79] $\sum_k p_k S(\rho_k) \leq S(\rho)$, where $\rho = \sum_k p_k \rho_k$, which follows immediately from equation (A.0.2) and Jensen's inequality. It also generalizes Shannon's classical version thereof, which states that observation on average reduces the entropy by an amount equal to the mutual information. The quantum entropy reduction cannot always be that large (which would give negative entropy after observing a qubit with $\mathbf{S} = \mathbf{I}$ and two bits of mutual information), but we have proved that it is never negative.

This generalization of the second law was conjectured by my supervisor Max Tegmark [64] to address the cosmological "entropy problem", which is to explain why our early universe had such low entropy, with its matter highly uniform rather than clumped into huge black holes. He addressed the entropy problem assuming unitary cosmology, using a subject, object and environment decomposition, and conjecturing generalized second law (S-theorems). Using the EOS decomposition and conjectured S-theorems, he argued that as long as inflation has occurred in a non-negligible fraction of the volume, almost all sentient observers will find themselves in a post- inflationary low-entropy Hubble volume, which solves the so-called inflationary entropy problem.

Results of this chapter complete the formalism of [64] for handling observed open systems and confirm Tegmark's argument about inflationary entropy problem. Furthermore, we have seen that quantum statistical mechanics still works flawlessly as long as we avoid sloppy talk of *the* density matrix and *the* entropy: we each have our own personal density matrix encoding everything we know about our object of interest, and we have simple formulas (Table 4.1) for how it changes under observation and decoherence, decreasing and increasing entropy.

Appendix A

Proof of Equation (4.5.24)

Let us first review three useful facts that we will use in our proof. As proven in [80], any Hermitean matrix $\mathbf{H}$ written in block form can be decomposed as

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\dagger & \mathbf{B} \end{bmatrix} = \mathbf{U} \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{U}^\dagger + \mathbf{V} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix} \mathbf{V}^\dagger \tag{A.0.1}$$

for some unitary matrices $\mathbf{U}$ and $\mathbf{V}$. Second, for any two Hermitean matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\boldsymbol{\lambda}(\mathbf{A}+\mathbf{B}) \prec \boldsymbol{\lambda}(\mathbf{A}) + \boldsymbol{\lambda}(\mathbf{B}). \tag{A.0.2}$$

This theorem was suggested and proved by Fan in 1949 and the proof is provided in [81] as Theorem 10.21. Finally, because the spectrum of a matrix is invariant under unitary transformations, we have

$$\boldsymbol{\lambda}(\mathbf{U}\mathbf{H}\mathbf{U}^\dagger) = \boldsymbol{\lambda}(\mathbf{H}) \tag{A.0.3}$$

for any Hermitean matrix $\mathbf{H}$ and any unitary matrix $\mathbf{U}$.

Combining these three facts, we obtain

$$\begin{aligned}
\boldsymbol{\lambda}\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\dagger & \mathbf{B} \end{pmatrix} &= \boldsymbol{\lambda}\left[\mathbf{U}\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^\dagger + \mathbf{V}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}\mathbf{V}^\dagger\right] \prec \\
&\prec \boldsymbol{\lambda}\left[\mathbf{U}\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^\dagger\right] + \boldsymbol{\lambda}\left[\mathbf{V}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}\mathbf{V}^\dagger\right] = \\
&= \boldsymbol{\lambda}\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \boldsymbol{\lambda}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix},
\end{aligned} \tag{A.0.4}$$

where the three logical steps use equations (A.0.1), (A.0.2) and (A.0.3), respectively.

Now consider a complete orthogonal set of Hermitean projection operators $\mathbf{P}_i$, $i = 1, ..., n$, satisfying the standard relations $\sum_{i=1}^n \mathbf{P}_i = \mathbf{I}$ and $\mathbf{P}_i\mathbf{P}_j = \delta_{ij}\mathbf{P}_i$. Since $\mathbf{P}_i^2 = \mathbf{P}_i$, all eigenvalues are either 0 or 1. Since all matrices $\mathbf{P}_i$ commute, there is basis where they are all diagonal, and where each matrix vanishes except for a block of ones somewhere along the diagonal. In this basis, $\mathbf{P}_i\mathbf{H}\mathbf{P}_i$ is simply $\mathbf{H}$ with all elements set to zero except for a corresponding square block. For example, for $n = 2$ we can write

$$\mathbf{P}_1\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\dagger & \mathbf{B} \end{pmatrix}\mathbf{P}_1 = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \mathbf{P}_2\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\dagger & \mathbf{B} \end{pmatrix}\mathbf{P}_2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}. \tag{A.0.5}$$

This means that we can rewrite the inequality (A.0.4) as

$$\boldsymbol{\lambda}(\mathbf{H}) \prec \sum_{i=1}^{2} \boldsymbol{\lambda}(\mathbf{P}_i\mathbf{H}\mathbf{P}_i) \tag{A.0.6}$$

for any Hermitean matrix $\mathbf{H}$. The sum of any of our two projection operators is a new projection operator, so by iterating equation (A.0.6), we can trivially generalize it to the case of arbitrary n:

$$\boldsymbol{\lambda}(\mathbf{H}) \prec \sum_{i=1}^{n} \boldsymbol{\lambda}(\mathbf{P}_i\mathbf{H}\mathbf{P}_i). \tag{A.0.7}$$

This concludes the proof of the left part of the double inequality (4.5.24). The right part follows directly from the equation (4.5.23) when making the choice

$$\mathbf{E}_{ij} \equiv \sum_{k=1}^{n} (\mathbf{P}_k)_{ii} (\mathbf{P}_k)_{jj} \tag{A.0.8}$$

in the basis where all projectors are diagonal. In other words, $\mathbf{E}$ is chosen to be the matrix with ones in all square blocks picked out by the projectors and zeroes everywhere else.

A.1 POVMs and Measurements

In this appendix, we derive the extent to which our entropy inequalities can be extended to interactions more general than probing. A Positive Operator Valued Measure (POVM) [73] is a projective measurement on the larger system that has the object as a subsystem. The additional quantum system is typically referred to as an ancilla (A). Specifically, a POVM is a mapping

$$\rho \to \{p_k, \rho^{(k)}\}, \tag{A.1.9}$$

where the resulting object state is $\rho^{(k)}$ with probability of outcome p_k.

A.1.1 General POVM

$\{p_k, \rho^{(k)}\}$ are given by

$$\rho^{(k)} = \frac{\chi_k(\rho)}{\operatorname{tr}\chi_k(\rho)}, \quad p_k = \operatorname{tr}\chi_k(\rho), \tag{A.1.10}$$

where

$$\chi_k(\rho) = \operatorname{tr}_A\left(\mathbf{P}_{OA}^{(k)}\left[\mathbf{U}\rho\otimes\rho_A\mathbf{U}^\dagger\right]\right). \tag{A.1.11}$$

Here A denotes the ancilla system in some initial state ρ_A and $\{\mathbf{P}_{OA}^{(k)}\}$ is orthonormal set of projectors acting on the object-ancilla system. Without loss of generality, we can take the ancilla state to be pure ($\rho_A = |a^*\rangle\langle a^*|$), because for a mixed ancilla state $\rho_A = \sum_i w_i|a_i\rangle\langle a_i|$, the purified state $|a^*\rangle \equiv \sum_i w_i^{1/2}|a_i\rangle\otimes|a_i\rangle$ defines the same POVM.

For a general POVM, neither of the two inequalities in the Equation (4.3.15) holds true, so we have no S-theorems for general POVMs. We will now prove this by providing POVM counterexamples that violate both the left (observation) and right (decoherence) inequalities in Equation (4.3.15).

- *Counterexample 1:* The object and ancilla are single qubits in a state $|0\rangle_O|0\rangle_A$

and the POVM is defined by $\mathbf{U} = \mathbf{I}$ and $\mathbf{P}_{AO}^{(k)} = |\Psi_k\rangle\langle\Psi_k|$, where

$$\begin{aligned} |\Psi_1\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\Psi_2\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} && \text{(A.1.12)} \\ |\Psi_3\rangle &= \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad |\Psi_4\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}. && \text{(A.1.13)} \end{aligned}$$

This gives

$$\begin{aligned} \rho^{(1)} &= \rho^{(2)} = \rho^{(3)} = \rho^{(4)} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, && \text{(A.1.14)} \\ p_1 &= p_2 = \frac{1}{2}, \qquad p_3 = p_4 = 0, && \text{(A.1.15)} \end{aligned}$$

so that the initial von Neumann entropy $S(\rho) = 0$ increases to a final entropy of 1 bit, violating the left (observation) part of the Equation (4.3.15).

- *Counterexample 2:* The object and ancilla are single qubits in initial states

$$\rho = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad \rho_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{A.1.16}$$

and $\mathbf{U}$ is the unitary two qubit operation that exchanges the states of these qubits (a swap gate)

$$\mathbf{U}\rho \otimes |0\rangle\langle 0|\mathbf{U}^\dagger = |0\rangle\langle 0| \otimes \rho. \tag{A.1.17}$$

The projections $\mathbf{P}_{OA}^{(k)}$ are

$$\begin{aligned} \mathbf{P}_{OA}^{(1)} &= \mathbf{I} \otimes |0\rangle\langle 0|, && \text{(A.1.18)} \\ \mathbf{P}_{OA}^{(2)} &= \mathbf{I} \otimes |1\rangle\langle 1|, && \text{(A.1.19)} \end{aligned}$$

giving

$$\begin{aligned}
\rho^{(1)} &= \rho^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\
p_1 &= p_2 = \frac{1}{2},
\end{aligned} \tag{A.1.20}$$

so that the initial entropy $S(\rho) = 1$ bit drops to zero, violating the right (decoherence) side of the inequality (4.3.15).

A.1.2 Purity-preserving POVM (PPPOVM)

We define a PPPOVM as a POVM that keeps a pure state pure, i.e., if ρ is a pure state, then $\rho^{(k)}$ is pure for all k. A POVM is purity preserving if and only if

$$\mathbf{P}_{OA}^{(k)} = \mathbf{I} \otimes |a_k\rangle\langle a_k|, \tag{A.1.21}$$

where $|a_k\rangle$ is an orthonormal basis of the ancilla system. To show this we, need to prove both the "if" and "only if" parts.

- **Part 1 ("if"):** For a pure object state $\rho = |\psi\rangle\langle\psi|$, the pure total state

  $$|\Psi\rangle^{OA} \equiv \mathbf{U}|\psi\rangle|a^*\rangle \tag{A.1.22}$$

  can always be decomposed as

  $$|\Psi\rangle^{OA} = \sum_k \lambda_k |\psi_k\rangle|a_k\rangle, \tag{A.1.23}$$

  where $|\psi_k\rangle$ are a normalized object states and λ_k are complex numbers. If $\mathbf{P}_{OA}^{(k)} = \mathbf{I} \otimes |a_k\rangle\langle a_k|$, then this decomposition gives

  $$\chi_k(\rho) = |\lambda_k|^2 |\psi_k\rangle\langle\psi_k|. \tag{A.1.24}$$

Since the state $\rho_k = |\psi_k\rangle\langle\psi_k|$ is pure for all k, we have shown that if $\mathbf{P}_{OA}^{(k)} = \mathbf{I} \otimes |a_k\rangle\langle a_k|$, then the POVM is purity preserving.

- **Part 2 ("only if"):** If the POVM is purity preserving, then we can write $\rho = |\psi\rangle\langle\psi|$ and $\rho_k = |\psi_k\rangle\langle\psi_k|$, so we have

$$\begin{aligned} \chi_k(\rho) &= \operatorname{tr}_A\left(\mathbf{P}_{OA}^{(k)}|\Psi\rangle^{OA}\langle\Psi|^{OA}\right) \\ &= p_k\,|\psi_k\rangle\langle\psi_k|\,. \end{aligned} \tag{A.1.25}$$

This means that after normalization, the state $\mathbf{P}_{OA}^{(k)}|\Psi\rangle^{OA}$ must be pure and separable for all k:

$$\mathbf{P}_{OA}^{(k)}|\Psi\rangle^{OA} = \lambda_k|\psi_k\rangle|a_k\rangle, \tag{A.1.26}$$

where $|\lambda_k|^2 = p_k$ and $|a_k\rangle$ is some state of the ancilla system. We can now express $\mathbf{P}_{OA}^{(k)}$ as

$$\mathbf{P}_{OA}^{(k)} = \mathbf{I} \otimes |a_k\rangle\langle a_k|\,, \tag{A.1.27}$$

and because these terms form an orthonormal set of projectors, we conclude that $|a_k\rangle$ form an orthonormal basis for ancilla. We have thus shown that if the POVM is purity preserving, then $\mathbf{P}_{OA}^{(k)} = \mathbf{I} \otimes |a_k\rangle\langle a_k|$ for some orthonormal basis $|a_k\rangle$.

In [76], Ozawa referred to PPPOVMs as "quasi-complete" and showed that the left inequality in Equation (4.3.15) is true if and only if a measurement is quasi-complete. He proved this for continuous measurements and only for von Neumann's definition of entropy. Our proof of left part of Equation (4.3.15) is readily generalized to PPPOVMs for our much more general definition of entropy, and without use of strong subadditivity. Here the ancilla plays the role of the subject and projections on the object-ancilla system is given by Equation (A.1.21), which are equivalent to the projectors $\mathbf{P}_k = \mathbf{I} \otimes |s_k\rangle\langle s_k|$ used in equations (4.5.24) and (4.5.26). Because the key equations (4.5.24) and (4.5.26) hold for arbitrary $\mathbf{U}$, the proof for the left (observation) part of Equation (4.3.15) can be immediately generalized to PPPOVMs. In contrary, the right part of Equation (4.3.15) does not hold for PPPOVM's, which can be seen using the counterexample 2 from the previous section. So only one of the S-theorems holds for PPPOVMs: observation always decreases average entropy, but decoherence does not always increase it.

A.1.3 Probing

Probing is a special case of a PPPOVM, where the role of the ancilla is played by either the subject or the environment, and the unitary dynamics $\mathbf{U}$ of the object-ancilla system leaves the object unchanged in some basis, merely changing the ancilla state in a way that depends on the object state $|o_i\rangle$:

$$\mathbf{U}|o_i\rangle|a^*\rangle = |o_i\rangle|a_i\rangle. \tag{A.1.28}$$

In this paper, we have shown that both inequalities in Equation (4.3.15) hold true for probing, so that observation always decreases average entropy and decoherence always increases it.

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